U.G.M.I.T, Rayagada

DEPARTMENT OF ELECTRICAL ENGINEERING

LECTURE NOTES ON

Control System Engineering

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2. ControlSystemsEngineering-S.Palani,TMH.
3. ControlSystems -N.K.Sinha,NewAgeInternational(P)Limited Publishers.
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UNIT-I

INTRODUCTION

A **control system** manages commands ,directs or regulates the behavior of other devices or system susing control loops. It can range from a single home heating controller using a thermost at controlling a domestic boiler to large Industrial control systems which are used for controlling processes or machines. A control system is a system, which provides the desired response by controlling the output. The following figure shows the simple block diagram of a control system.

Examples–Traffic lights control system,washing machine

Traffic lights control system is an example of control system. Here, a sequence of input signal is applied to this control system and the output is one of the three lights that will be on for some duration of time. During this time , the other two lights will be off. Based on the traffic study at a particular junction, the on and off times of the lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights control system operates on time basis.

Classification of Control Systems

Based on some parameters, we can classifythecontrolsystemsintothefollowing ways.

ContinuoustimeandDiscrete-timeControl Systems

1. Control Systems can be classified as continuoustime control systems and discrete time control systems based on the **type of the signal** used.

2. In**continuoustime**controlsystems,allthesignalsarecontinuousintime.But,in **discrete time** control systems, there exists one or more discrete time signals.

SISO and MIMO Control Systems

3. ControlSystemscanbeclassifiedasSISOcontrolsystemsandMIMOcontrolsystems based on the **number of inputs and outputs** present.

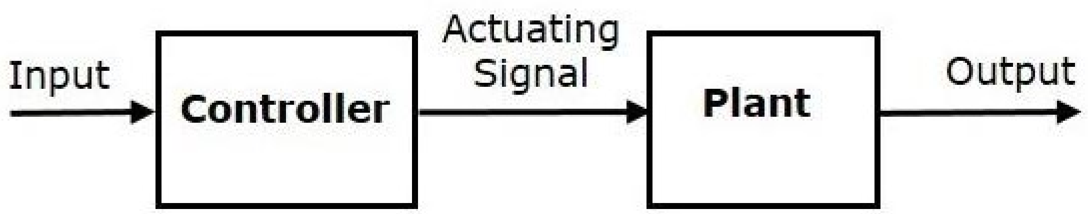
4. **SISO**(SingleInputandSingleOutput)controlsystemshaveoneinputandoneoutput. Whereas,**MIMO**(Multiple Inputs and Multiple Outputs) controlsystems have more than one input and more than one output.

OpenLoopandClosedLoopControlSystems

ControlSystemscanbeclassifiedasopenloopcontrolsystemsandclosedloopcontrol systems based on the **feedback path**.

In**open loop control systems**, output is not fed-back to the input. So, the control action is independent of the desired output.

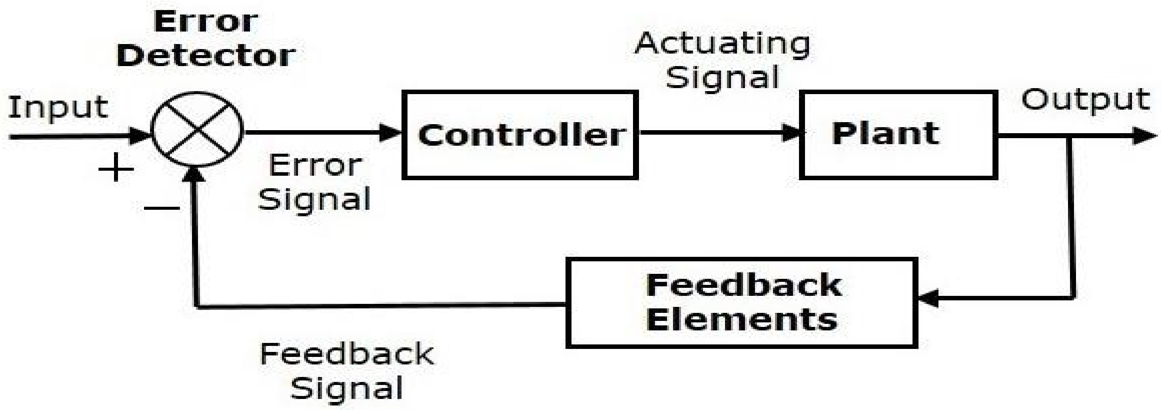
Thefollowingfigureshowstheblockdiagram oftheopenloopcontrolsystem.



Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signalis givenas aninput toa plant or process whichis tobe controlled. So,the plant produces an output, which is controlled. The traffic lights control system which we discussed earlier is an example of an open loop control system.

In**closed loop control systems**, output is fed back to the input. So, the control action is dependent on the desired output.

The following figure shows the block diagram of negative feedback closed loop control system.



The error detector produces anerror signal, whichis the difference betweenthe input and thefeedbacksignal. This feedbacksignalis obtainedfromtheblock (feedbackelements)by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

So,thecontrollerproducesanactuatingsignalwhichcontrolstheplant.Inthiscombination, the output of the control system is adjusted automatically till we get the desired response. Hence,theclosedloopcontrolsystemsarealsocalledtheautomaticcontrolsystems.Traffic lightscontrolsystemhavingsensorattheinputisanexampleofaclosedloopcontrolsystem.

The differences betweenthe open loopandtheclosedloopcontrolsystemsarementioned in the following table.

Open Loop Control Systems	Closed Loop Control Systems
Control action is independent of the desired output.	Control action is dependent of the desired output.
Feedback path is not present.	Feedback path is present.
These are also called as **non-feedback control systems**.	These are also called as **feedback control systems**.
Easy to design.	Difficult to design.
These are economical.	These are costlier.
Inaccurate.	Accurate.

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then it is known as **feedback**. Feedback plays an important role in order to improve the performance of the control systems. In this chapter, let us discuss the types of feedback & effects of feedback.

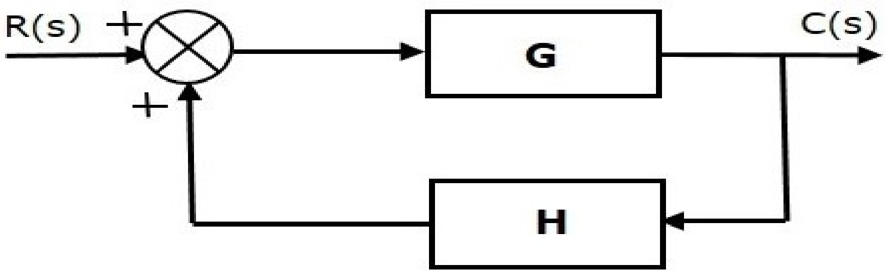
TypesofFeedback

Thereare twotypesoffeedback−

5. Positivefeedback
6. Negative feedback

PositiveFeedback

Thepositivefeedbackaddsthereferenceinput,R(s)R(s)andfeedbackoutput.Thefollowing figure shows the block diagram of **positive feedback control system**



he concept of transfer function will be discussed in later chapters. For the time being, consider the transfer function of positive feedback control system is,

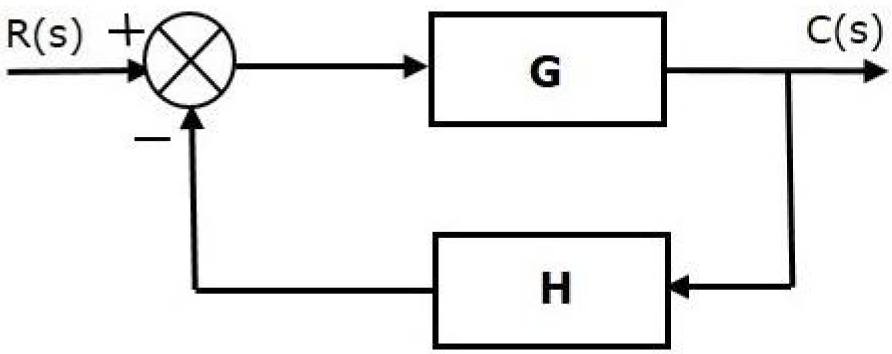
$$T = \frac{G}{1-GH} \qquad \text{(Equation 1)}$$

Where,

7. **T**isthetransferfunctionoroverallgainofpositivefeedbackcontrolsystem.
8. **G**istheopen loopgain,whichisfunctionoffrequency.
9. **H**isthegainoffeedbackpath,whichisfunctionoffrequency.

NegativeFeedback

Negative feedback reduces the error between the reference input, R(s)R(s)and system output. The following figure shows the block diagram of the **negative feedback control system**.



Transferfunctionofnegativefeedbackcontrolsystem is,

$$T = \frac{G}{1+GH} \qquad \text{(Equation 2)}$$

Where,

10. **T**isthetransferfunctionoroverallgainofnegativefeedbackcontrolsystem.

11. **G**istheopen loopgain,whichisfunctionoffrequency.

12. **H**isthegainoffeedbackpath,whichisfunctionoffrequency.

Thederivation ofthe abovetransferfunctionispresentinlaterchapters.

EffectsofFeedback

Letusnowunderstandtheeffectsoffeedback.

EffectofFeedbackonOverall Gain

13. From Equation 2, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and (1+GH). So, the overall gain may increase or decrease depending on the value of (1+GH).

14. Ifthe value of(1+GH) is less than 1,then the overallgain increases. Inthis case,'GH' value is negative because the gain of the feedback path is negative.

15. Ifthe value of (1+GH) is greater than 1, then the overall gain decreases. Inthis case, 'GH' value is positive because the gain of the feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, the feedback will increase the overall gain of the system in one frequency range and decrease in the other frequency range.

EffectofFeedbackon Sensitivity

Sensitivity oftheoverallgainofnegativefeedbackclosedloopcontrolsystem(**T**)tothe variation in open loop gain (**G**) is defined as

$$S_G^T = \frac{\frac{\partial T}{T}}{\frac{\partial G}{G}} = \frac{Percentage\ change\ in\ T}{Percentage\ change\ in\ G} \qquad \text{(Equation 3)}$$

Where, **∂T** is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} \qquad \text{(Equation 4)}$$

Do partial differentiation with respect to G on both sides of Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \qquad \text{(Equation 5)}$$

From Equation 2, you will get

$$\frac{G}{T} = 1 + GH \qquad \text{(Equation 6)}$$

Substitute Equation 5 and Equation 6 in Equation 4.

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So,we got the**sensitivity** ofthe overallgain of closed loopcontrolsystem as the reciprocal of (1+GH). So, Sensitivity may increase or decrease depending on the value of (1+GH).

16. Ifthevalue of (1+GH) is lessthan1,then sensitivityincreases.Inthiscase,'GH'value is negative because the gain of feedback path is negative.

17. If the value of (1+GH) is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path is positive.

Ingeneral,'G'and'H'arefunctionsoffrequency.So,feedbackwillincreasethesensitivityof the system gain in one frequency range and decrease in the other frequency range. Therefore,wehave tochoosethevaluesof'GH' insucha waythatthe systemisinsensitive or less sensitive to parameter variations.

EffectofFeedbackon Stability

18. A system is saidtobe stable, if its output is under control. Otherwise, it is said tobe unstable.

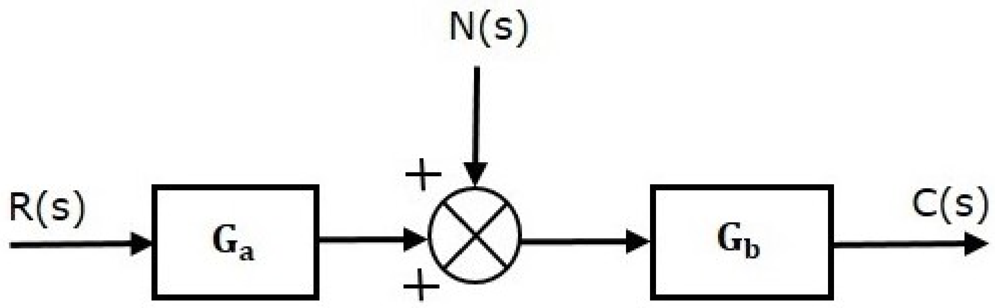
19. In Equation 2, if the denominator value is zero (i.e.,GH = -1), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore, we have to properly choose the feedback in order to make the control systemstable.

EffectofFeedbackon Noise

Toknowtheeffectoffeedbackonnoise,letuscomparethetransferfunctionrelationswith and without feedback due to noise signal alone.

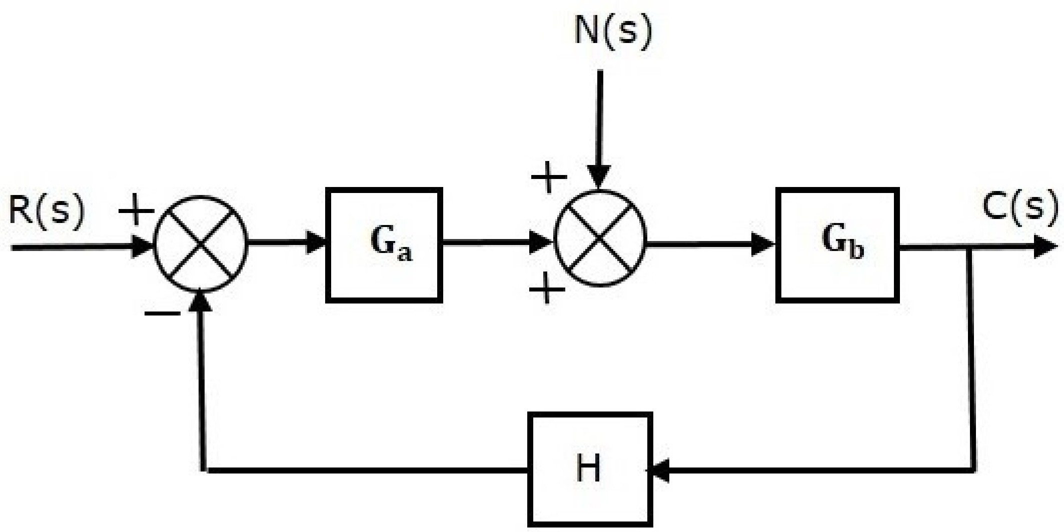
Consideran **openloopcontrolsystem** withnoisesignalasshownbelow.



The **open loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = G_b \qquad \text{(Equation 7)}$$

It is obtained by making the other input $R(s)$ equal to zero.



The **closed loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = \frac{G_b}{1+G_aG_bH} \qquad \text{(Equation 8)}$$

It is obtained by making the other input $R(s)$ equal to zero.

Compare Equation 7 and Equation 8,

In the closed loop control system, the gain due to noise signal is decreased by a factor of $(1+G_aG_bH)$ provided that the term $(1+G_aG_bH)$ is greater than one.

Thecontrolsystemscanberepresentedwithasetofmathematicalequationsknown as **mathematicalmodel**.Thesemodelsareusefulforanalysisanddesignofcontrolsystems. Analysisofcontrolsystemmeansfindingtheoutputwhenweknowtheinputand mathematicalmodel.Designofcontrolsystemmeansfindingthemathematicalmodelwhen we know the input and the output.

Thefollowingmathematicalmodelsaremostly used.

20. Differentialequationmodel
21. Transferfunctionmodel
22. Statespacemodel

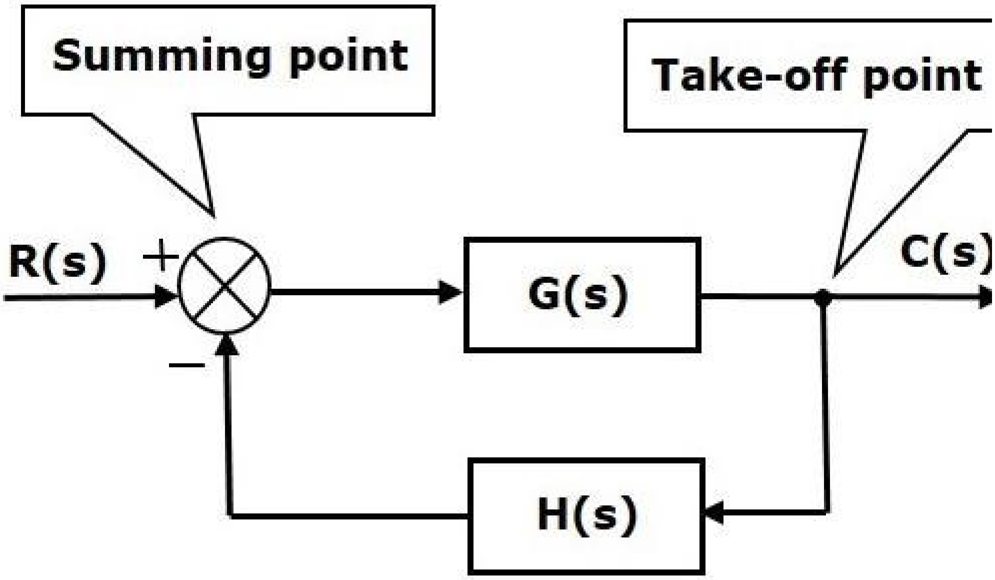
TRANSFERFUNCTIONREPRESENTATION

BlockDiagrams

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

BasicElementsofBlockDiagram

Thebasicelementsofablockdiagramareablock,thesummingpointandthetake-offpoint. Let us considerthe block diagram ofa closedloop controlsystem as showninthe following figure to identify these elements.



The above block diagram consists oftwoblocks having transferfunctions G(s)andH(s). It is alsohaving onesumming point and onetake-off point. Arrows indicatethe direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

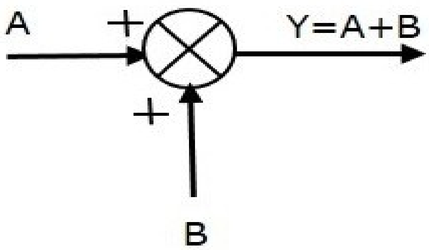
Transfer Function, $$G(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow Y(s) = G(s)X(s)$$

Summing Point

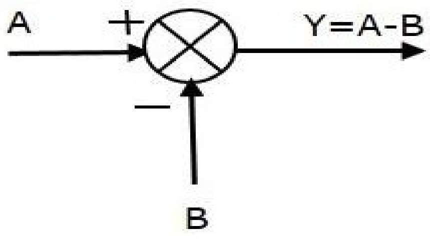
The summing point is representedwitha circle having cross (X)inside it. It has twoormore inputs and single output. It produces the algebraic sum of the inputs. It also performs the summationorsubtractionorcombinationofsummationandsubtractionoftheinputsbased on the polarity of the inputs. Let us see these three operations one by one.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here,the inputsAandBhaveapositivesign.So,thesummingpoint producestheoutput,Y as **sum of A and B** i.e. = A + B.



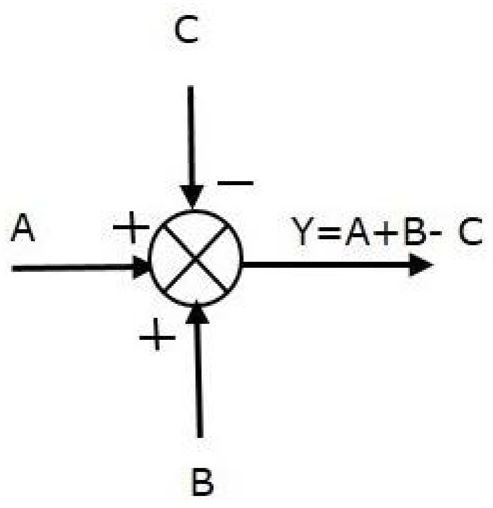
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y**as the**difference of A and B** i.e

Y=A+(-B)=A-B.


A
+
Y=A-B
−
B

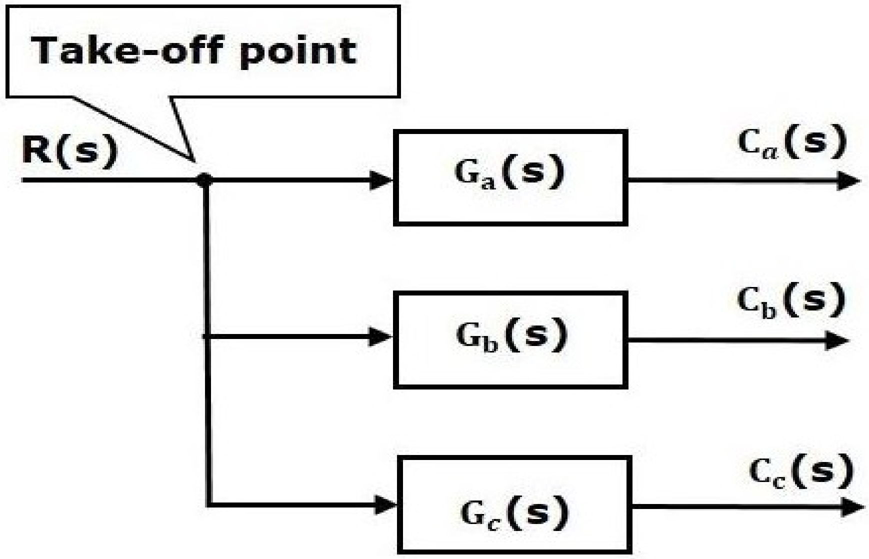
Thefollowingfigureshowsthesummingpointwiththreeinputs(A,B,C)andoneoutput(Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

Y=A+B+(−C)=A+B−C.

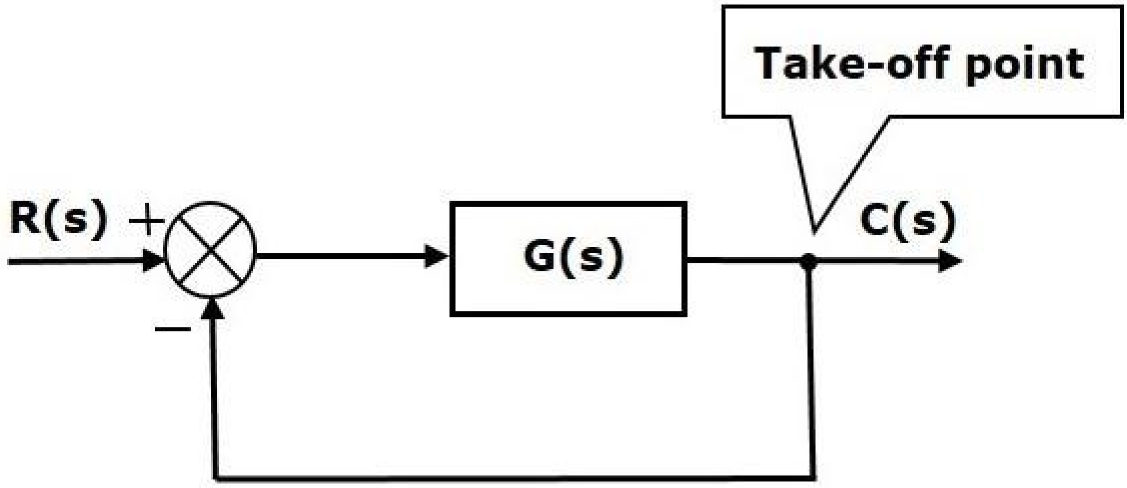


Take-offPoint

The take-offpoint is a point from which the same input signal can be passed throughmore than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks,summing points.In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.


Take-off point
R(s)
$G_a(s)$
$C_a(s)$
$G_b(s)$
$C_b(s)$
$G_c(s)$
$C_c(s)$

In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.



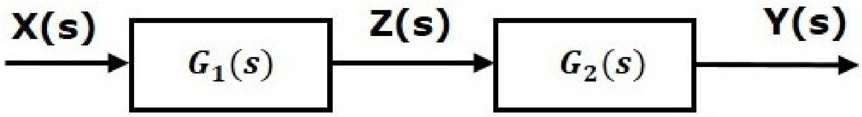
Block diagram algebra is nothing but the algebra involved with the basic elements of theblock diagram. This algebra deals with the pictorial representation of algebraic equations.

BasicConnectionsforBlocks

Therearethreebasictypesofconnectionsbetweentwoblocks.

SeriesConnection

Seriesconnectionisalsocalled **cascadeconnection**.Inthefollowingfigure,twoblocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in series.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

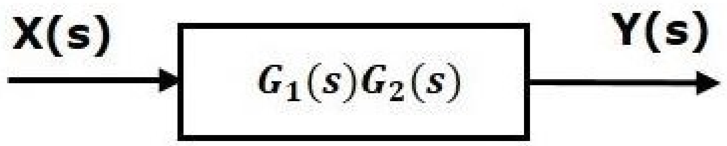
Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$. Where, $G(s) = G_1(s)G_2(s)$.

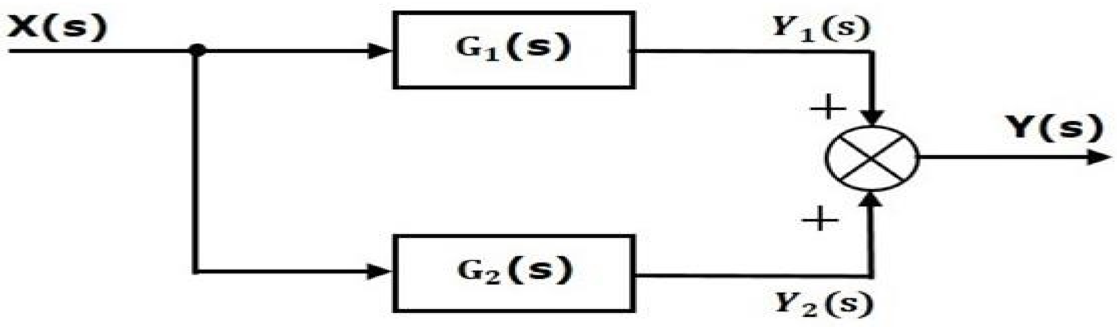
That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent series connection of ‘n’ blocks witha single block. The transfer function of this single block is the product of the transfer functions of all those ‘n’ blocks.

ParallelConnection

Theblockswhichareconnectedin**parallel**willhavethe**sameinput**.Inthefollowingfigure, two blocks having transfer functions G1(s)G1(s) andG2(s)G2(s)are connected in parallel. The outputs of these two blocks are connected to the summing point.



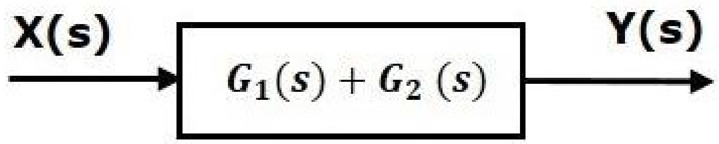
$$Y(s) = Y_1(s) + Y_2(s)$$

$Y_1(s) = G_1(s)X(s)$ and $Y_2(s) = G_2(s)X(s)$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

$$G(s) = G_1(s) + G_2(s).$$

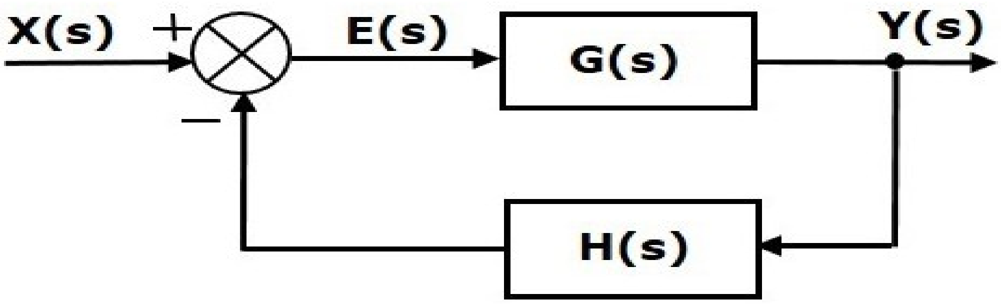
Thatmeanswe can representthe **parallelconnection** oftwoblockswitha single block.The transferfunctionofthissingleblockisthe **sumofthetransferfunctions** ofthosetwoblocks. The equivalent block diagram is shown below.



Similarly,youcanrepresentparallelconnectionof‘n’blockswithasingleblock.Thetransfer function of this single block is the algebraic sum of the transfer functions of all those ‘n’ blocks.

FeedbackConnection

As we discussed inprevious chapters,there are twotypes of **feedback**— positive feedback andnegativefeedback.The followingfigure showsnegativefeedback controlsystem.Here, two



The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is -

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$

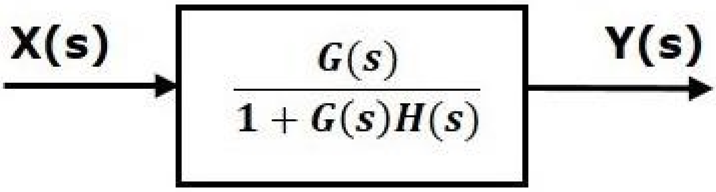
$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

blocks having transfer functions G(s)G(s) and H(s)H(s) form a closed loop.

Therefore,thenegativefeedbackclosedlooptransferfunctionis:

$$\frac{G(s)}{1+G(s)H(s)}$$

This means we can represent the negative feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.



Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e.,

$$\frac{G(s)}{1-G(s)H(s)}$$

BlockDiagramAlgebraforSummingPoints

Thereare twopossibilitiesofshiftingsummingpointswithrespecttoblocks−

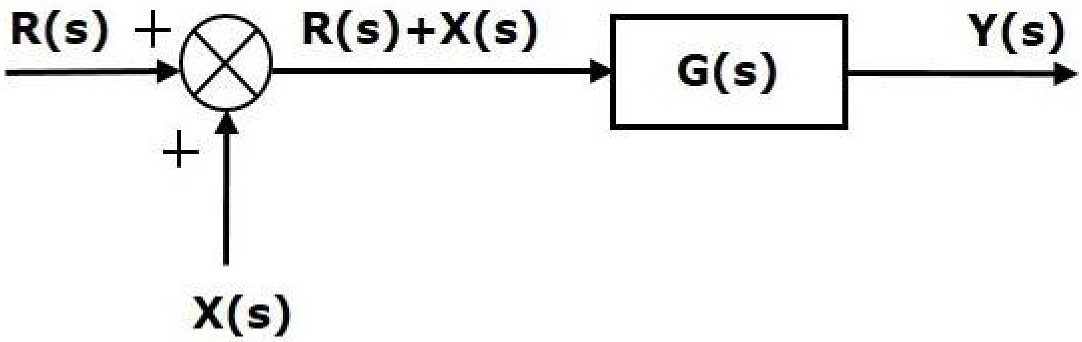
1. Shiftingsummingpointaftertheblock
2. Shiftingsummingpointbeforetheblock

Let us now see what kind of arrangements need to be done in the above two cases one by one.

ShiftingtheSummingPointbeforeaBlocktoafteraBlock

Considertheblockdiagramshowninthefollowingfigure.Here,thesummingpointispresent before the block.

$\{R(s) + X(s)\}$.



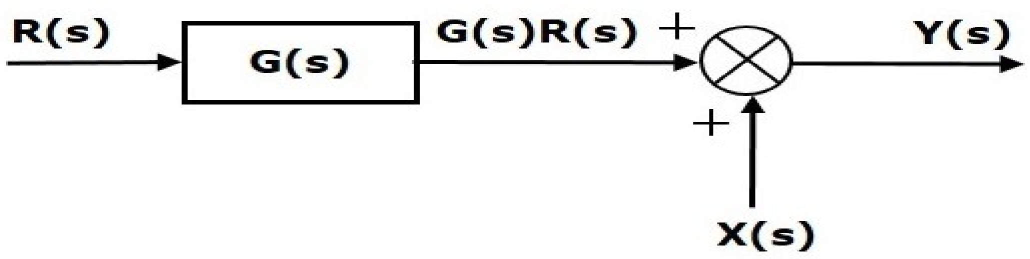
Summing point has two inputs $R(s)$ and $X(s)$

TheoutputofSummingpointis

So, the input to the block $G(s)$ is $\{R(s) + X(s)\}$ and the output of it is –

$$Y(s) = G(s)\{R(s) + X(s)\}$$

$$\Rightarrow Y(s) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 1)}$$



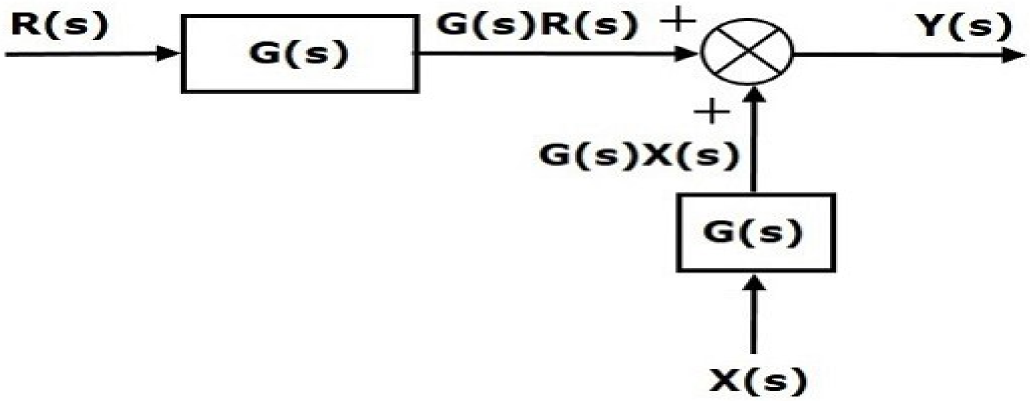
Output of the block $G(s)$ is $G(s)R(s)$.

The output of the summing point is

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 2)}$$

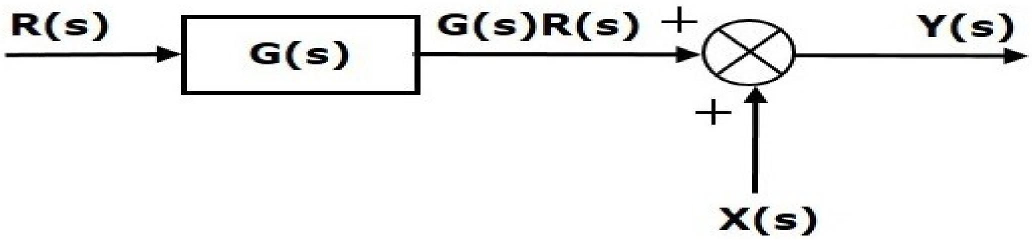
CompareEquation1andEquation2.

Thefirstterm‘G(s)R(s)’‘G(s)R(s)’issameinboththeequations.But,thereisdifferenceinthe secondterm.Inordertogetthesecondtermalsosame,werequireonemoreblockG(s)G(s). Itishavingtheinput X(s)X(s)andthe output ofthisblock isgivenasinputtosummingpoint instead of X(s)X(s). This block diagram is shown in the following figure.



Shifting Summing Point Before the Block

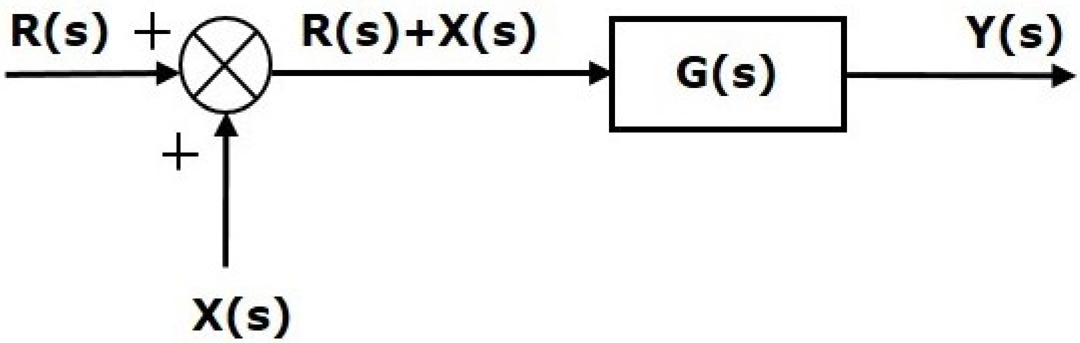
Consider the block diagram shown in the following figure. Here, the summing point is present after the block.



Output of this block diagram is -

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 3)}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.

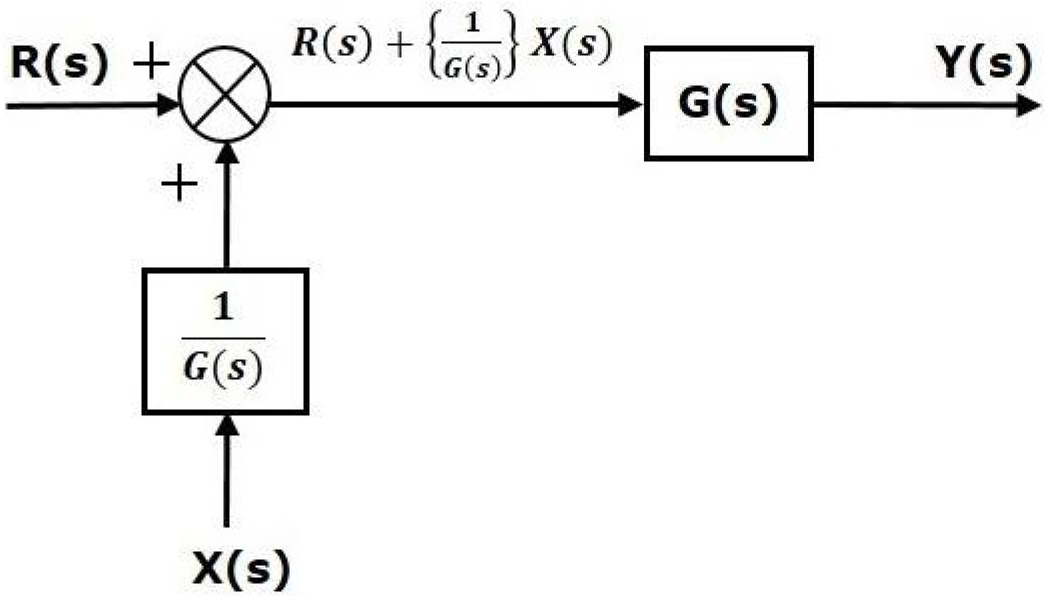


Output of this block diagram is -

$$Y(S) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 4)}$$

CompareEquation3andEquation4,

Thefirstterm‘G(s)R(s)’issameinbothequations.But,thereisdifferenceinthesecondterm. Inordertogetthesecondtermalsosame,werequireonemoreblock1/G(s).Itishavingthe inputX(s)andtheoutputofthisblockisgivenasinputtosummingpointinsteadof X(s).This block diagram is shown in the following figure.



BlockDiagramAlgebraforTake-offPoints

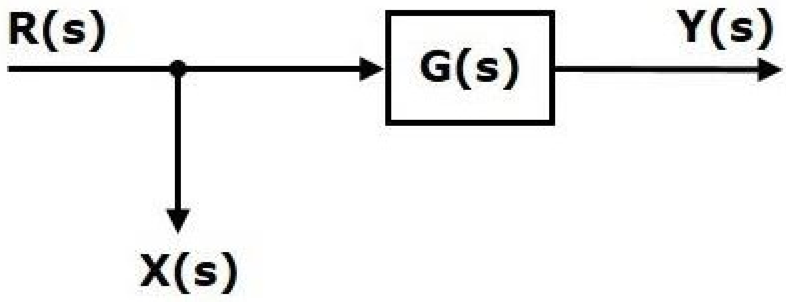
Thereare twopossibilitiesofshiftingthetake-offpointswithrespecttoblocks−

1. Shiftingtake-offpointafterthe block
2. Shiftingtake-offpoint beforetheblock

Letusnowseewhatkindofarrangementsistobedoneintheabovetwocases,onebyone.

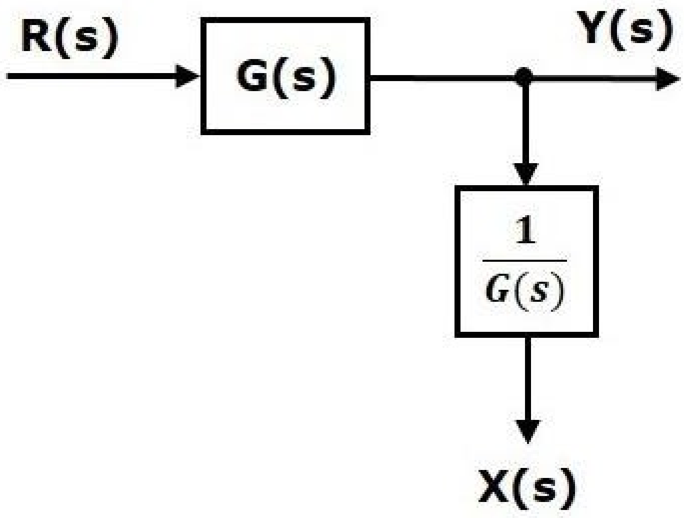
Shifting a Take-off Point form a Position before a Block to a position after the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.

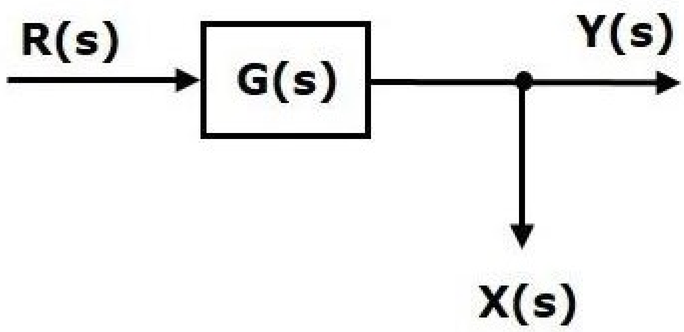


Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

When youshift the take-off point afterthe block, the output Y(s) will be same. But,there is differenceinX(s)value.So,inordertogetthesameX(s)value,werequireonemore block 1/G(s). It is having the input Y(s) and the output is X(s) this block diagram is shown in the following figure.

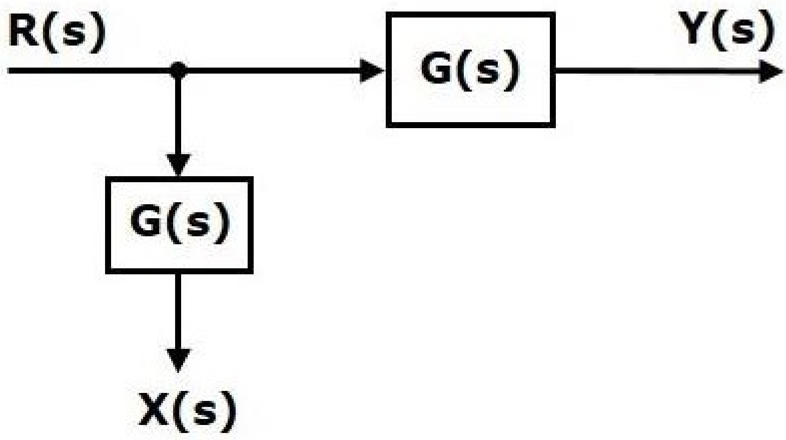


Shifting Take-off Point from a Position after a Block to a position before the Block Considertheblockdiagramshowninthefollowingfigure.Here, thetake-offpointispresent after the block.



Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block,the output Y(s) will be same. But, there isdifferenceinX(s)value.So,inordertogetsame X(s)value,werequireonemoreblock G(s) ItishavingtheinputR(s)andtheoutputisX(s).Thisblockdiagramisshowninthefollowing figure.



The concepts discussed in theprevious chapterare helpful for reducing (simplifying) theblock diagrams.

BlockDiagramReductionRules

Followtheserulesforsimplifying(reducing)theblockdiagram,whichishavingmanyblocks, summing points and take-off points.

3. **Rule1**– Checkfortheblocksconnected inseriesandsimplify.
4. **Rule2**–Checkfortheblocksconnected inparallelandsimplify.
5. **Rule3**–Checkfortheblocksconnectedinfeedbackloopand simplify.
6. **Rule4**–Ifthereisdifficultywithtake-offpointwhilesimplifying,shiftittowardsright.
7. **Rule5**–Ifthereisdifficultywithsummingpointwhilesimplifying,shiftittowardsleft.
8. **Rule 6** – Repeat the above steps till you get the simplified form, i.e., single block.

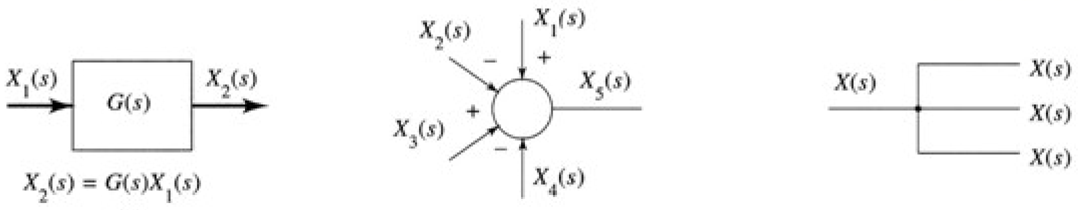
Note–Thetransferfunctionpresentinthissingleblockisthetransferfunctionoftheoverall block diagram.

Note–Followthesestepsinordertocalculatethetransfer functionoftheblock diagram having multiple inputs.

9. **Step1**–Findthetransferfunctionofblockdiagrambyconsideringoneinputata time and make the remaining inputs as zero.
10. **Step2**–Repeatstep1 forremaininginputs.
11. **Step3**–Gettheoveralltransferfunctionbyaddingallthosetransferfunctions.

Theblockdiagramreductionprocesstakesmoretimeforcomplicatedsystemsbecause;we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

BlockDiagram Reduction- Summary



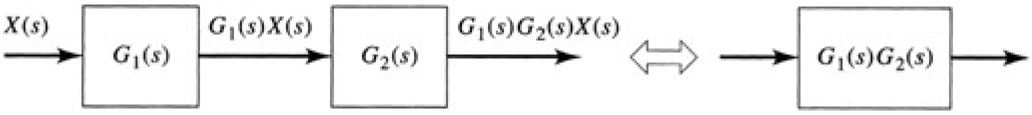
$X_2(s) = G(s)X_1(s)$

(a)
block

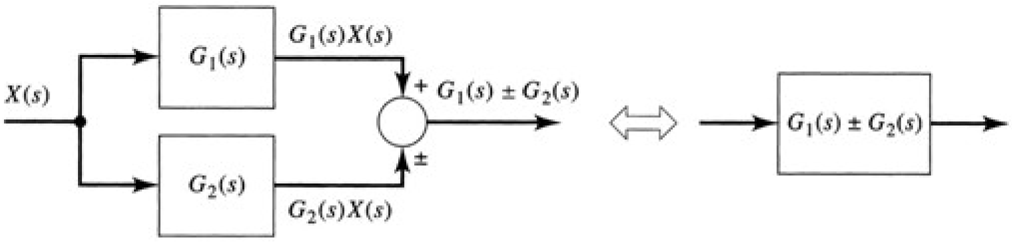
$X_5(s) = X_1(s) - X_2(s) + X_3(s) - X_4(s)$

(b) *summer*

(c)
pickoff point

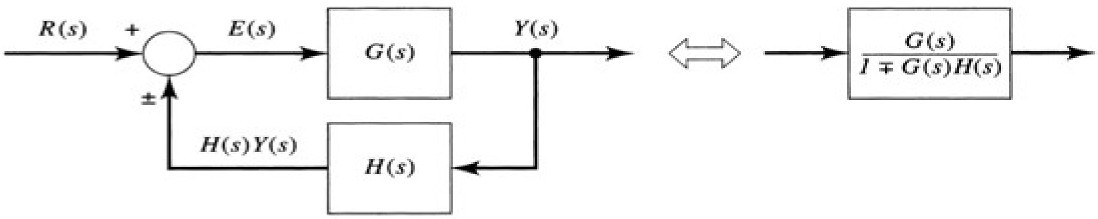


(a)



(b)

Automatic control

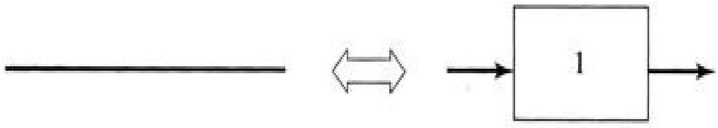


$$Y(s) = G(s)E(s)$$

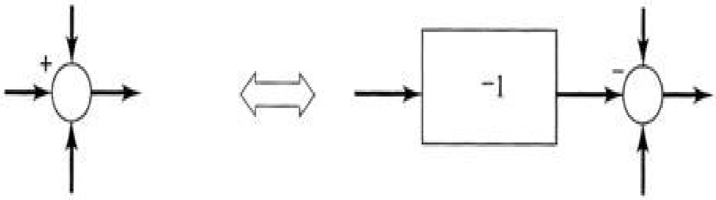
$$E(s) = R(s) \pm H(s)Y(s)$$

$$Y(s) = G(s)[R(s) \pm H(s)Y(s)] = G(s)R(s) \pm G(s)H(s)Y(s)$$

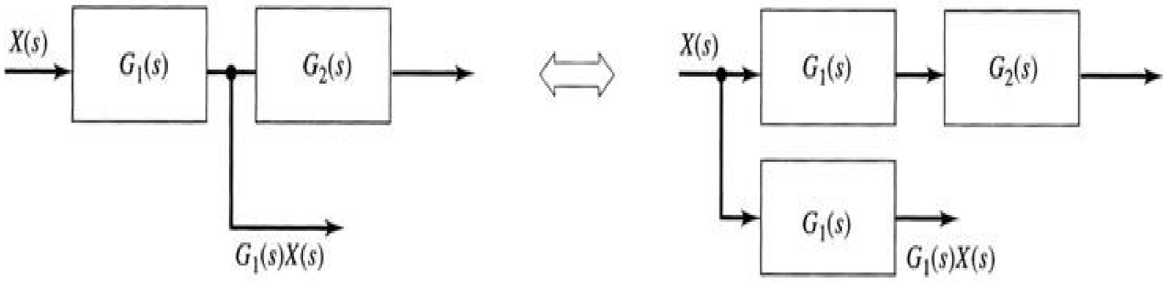
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}$$



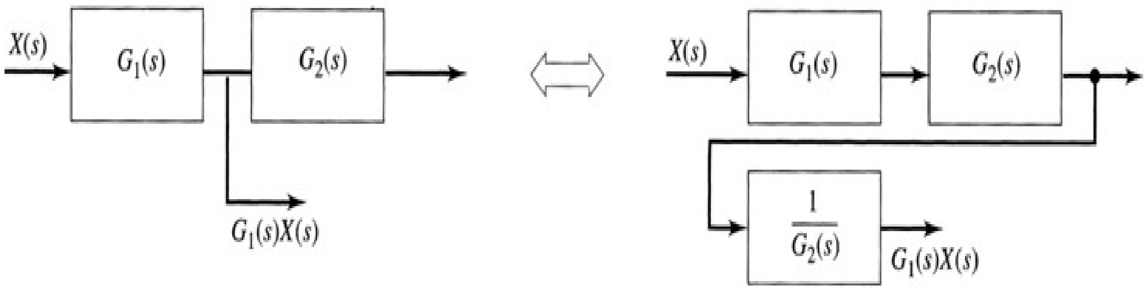
(*a*) Insertion or removal of unity gain



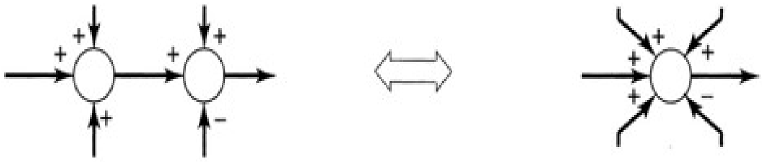
(*b*) Changing a summer sign



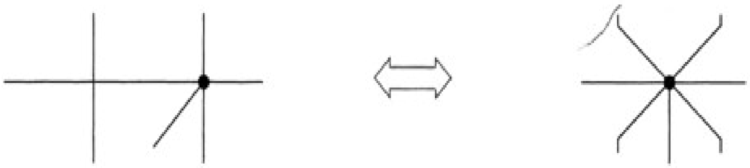
(*c*) Moving a pickoff point back



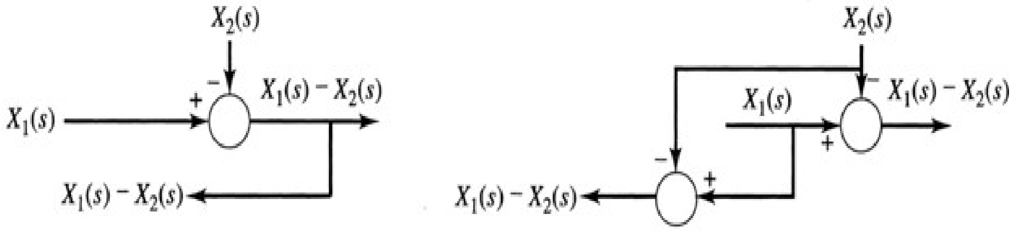
(*d*) Moving a pickoff point forward



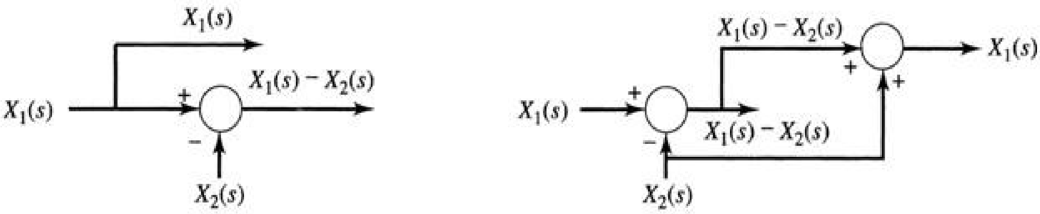
(e) Combining or expanding summations



(f) Combining or expanding junctions



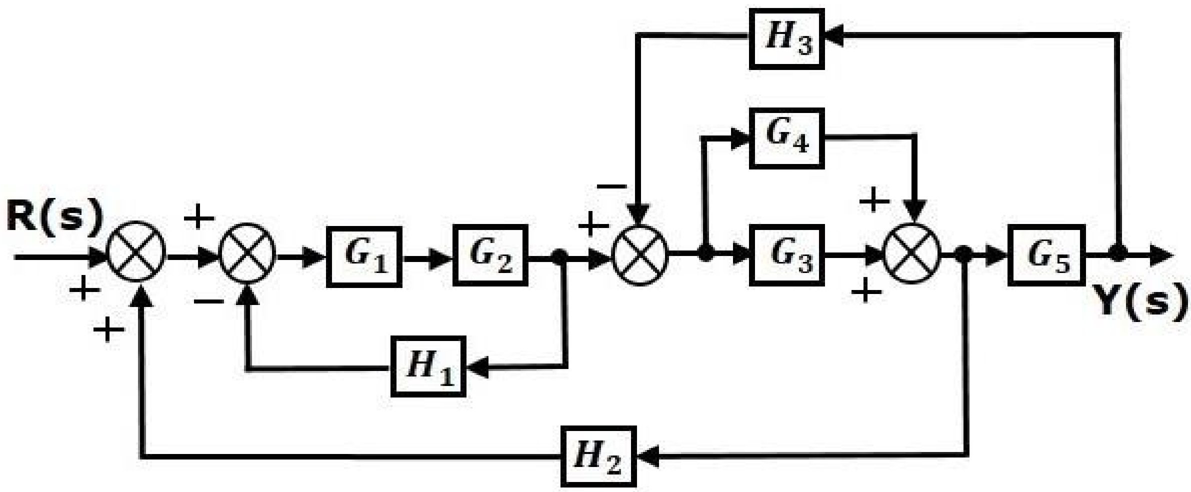
(g) Moving a pickoff point behind a summation



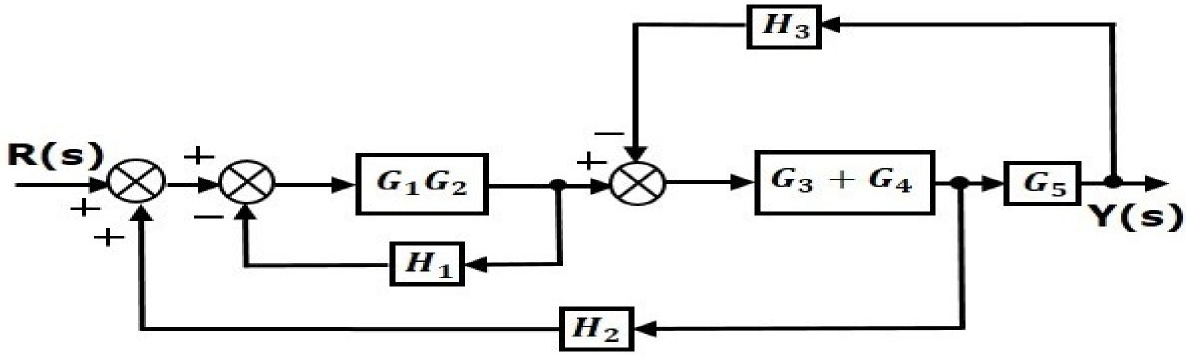
(h) Moving a pickoff point forward of a summation

Examples:

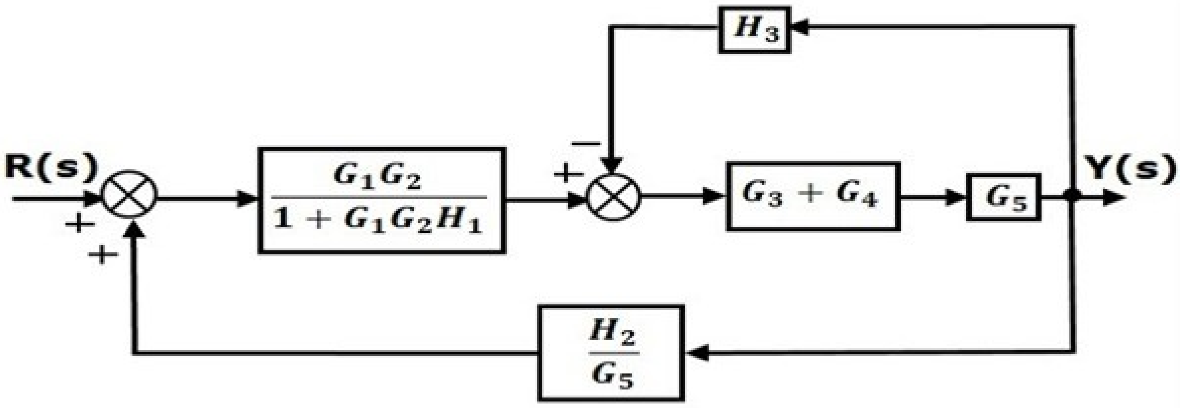
1. Considertheblockdiagramshowninthe following figure.Let ussimplify(reduce)this block diagram using the block diagram reduction rules.



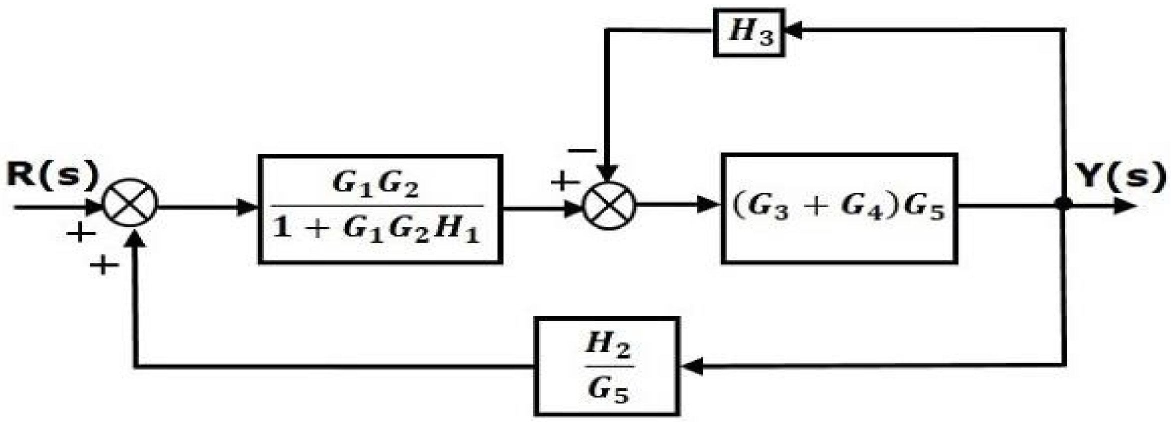
Step 1 − Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks G_3 and G_4. The modified block diagram is shown in the following figure.



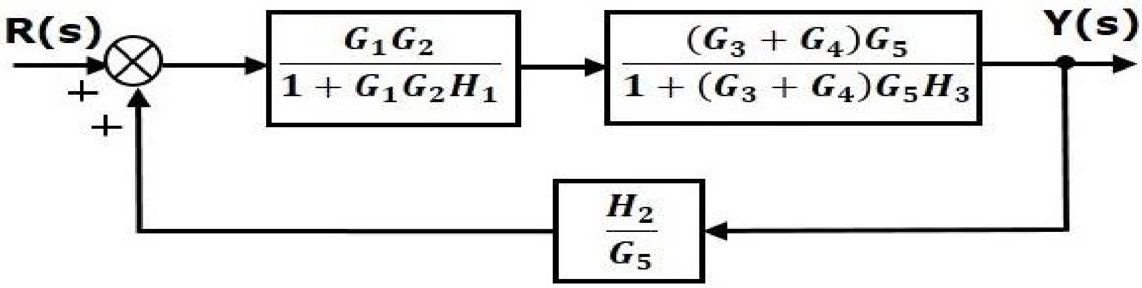
Step 2 – Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.



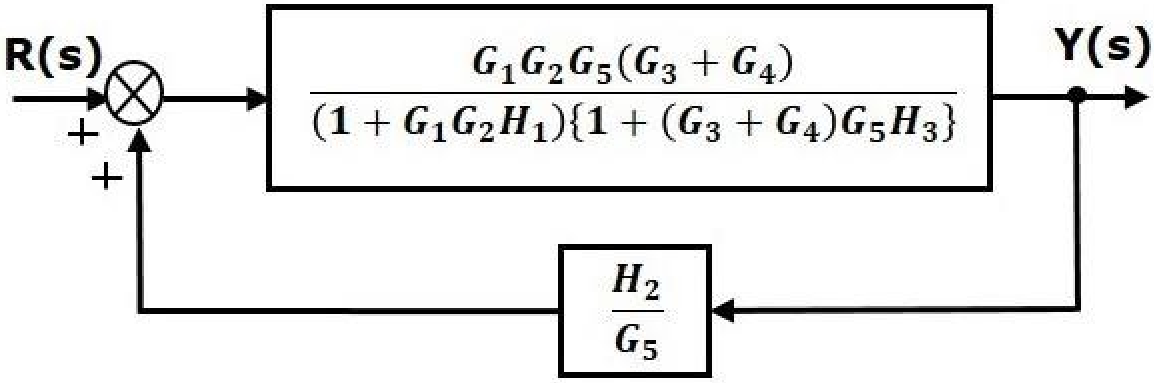
Step 3 – Use Rule 1 for blocks $(G_3 + G_4)$ and G_5. The modified block diagram is shown in the following figure.



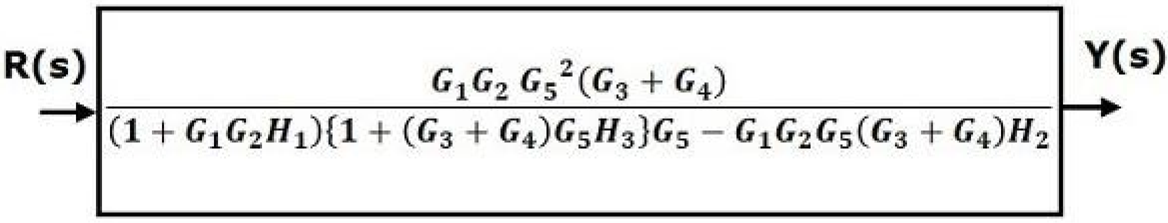
Step 4 – Use Rule 3 for blocks $(G_3 + G_4)G_5$ and H_3. The modified block diagram is shown in the following figure.



Step 5 – Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



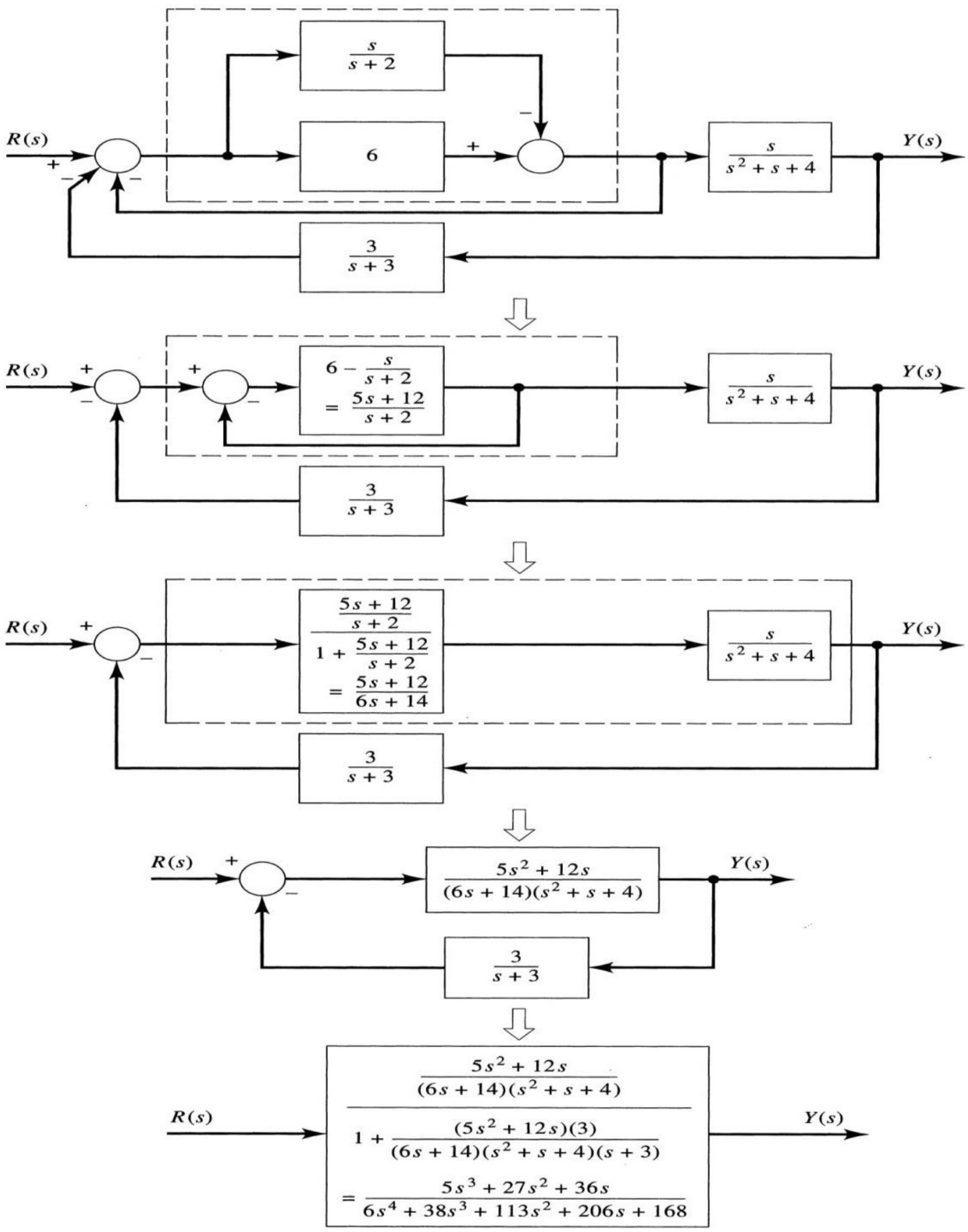
Step 6 – Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



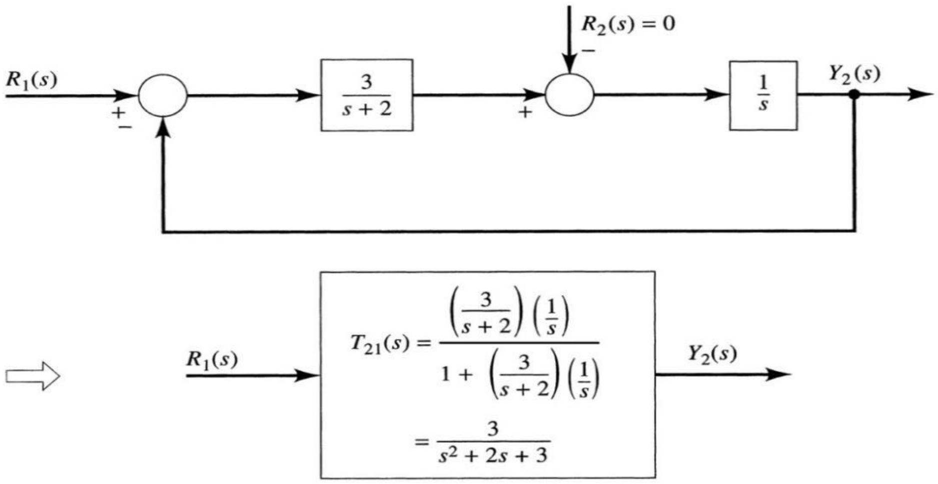
Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

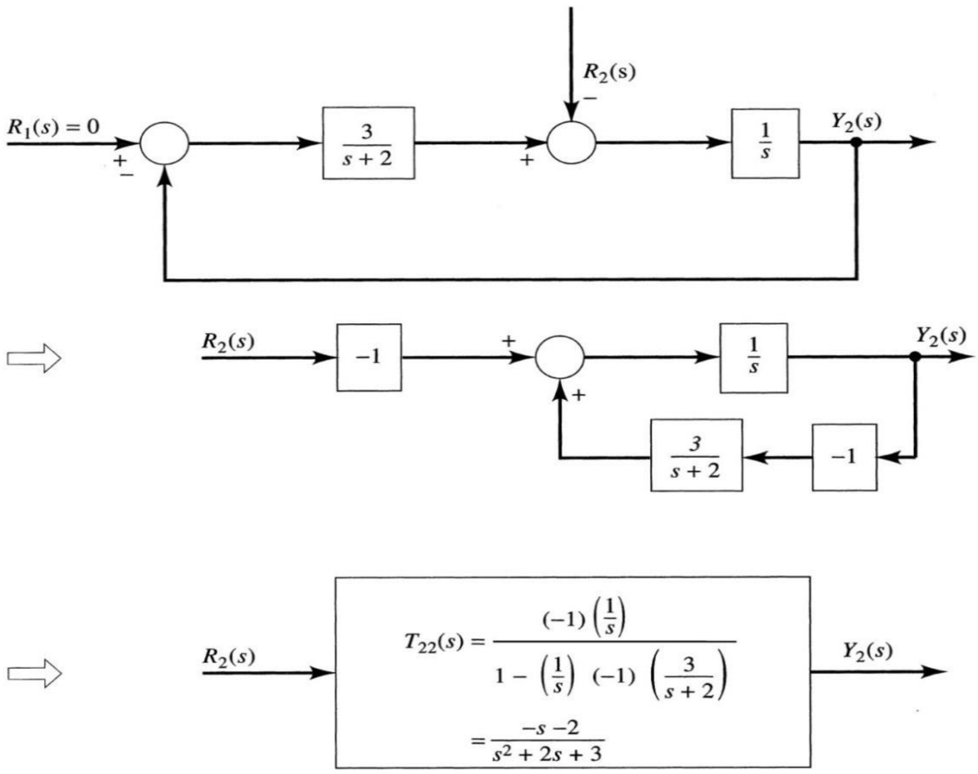
2.DeterminethetransferfunctionY(s)/R(s).



3. Determinethetransferfunction$Y_2(s)/R_1(s)$.


$R_2(s) = 0$
$R_1(s)$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_1(s)$
$T_{21}(s) = \dfrac{\left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)}{1 + \left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)} = \dfrac{3}{s^2+2s+3}$
$Y_2(s)$

(a)


$R_2(s)$
$R_1(s) = 0$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_2(s)$
-1
$\frac{1}{s}$
$Y_2(s)$
$\frac{3}{s+2}$
-1
$R_2(s)$
$T_{22}(s) = \dfrac{(-1)\left(\frac{1}{s}\right)}{1 - \left(\frac{1}{s}\right)(-1)\left(\frac{3}{s+2}\right)} = \dfrac{-s-2}{s^2+2s+3}$
$Y_2(s)$

(b)

SignalFlowGraph

Signalflowgraphisagraphicalrepresentationofalgebraicequations.Inthischapter,letus discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

BasicElementsofSignalFlowGraph

Nodesandbranchesarethebasicelementsofsignalflowgraph.

Node

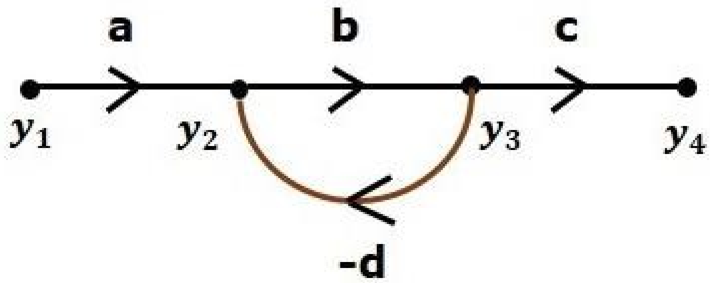
Nodeisapointwhichrepresentseitheravariableorasignal.Therearethreetypesofnodes

1. inputnode,outputnodeandmixednode.

    1. **InputNode**−Itisanode,whichhasonlyoutgoingbranches.
    2. **OutputNode**−Itisanode,which hasonlyincomingbranches.
    3. **MixedNode**−Itisanode,whichhasbothincomingandoutgoingbranches.

Example

Letusconsiderthefollowingsignalflowgraphtoidentifythesenodes.



- The **nodes** present in this signal flow graph are $\mathbf{y_1}$, $\mathbf{y_2}$, $\mathbf{y_3}$ and $\mathbf{y_4}$.
- $\mathbf{y_1}$ and $\mathbf{y_4}$ are the **input node** and **output node** respectively.
- $\mathbf{y_2}$ and $\mathbf{y_3}$ are **mixed nodes**.

Branch

Branchisalinesegmentwhichjoinstwonodes.Ithasboth **gain**and **direction**.Forexample, there are fourbranchesin the above signal flowgraph. Thesebrancheshave**gains**of**a, b,c** and **-d**.

ConstructionofSignalFlowGraph

Letusconstructasignalflowgraphbyconsideringthefollowingalgebraicequations−

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

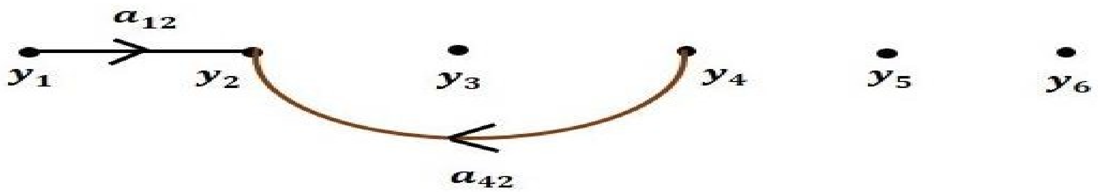
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

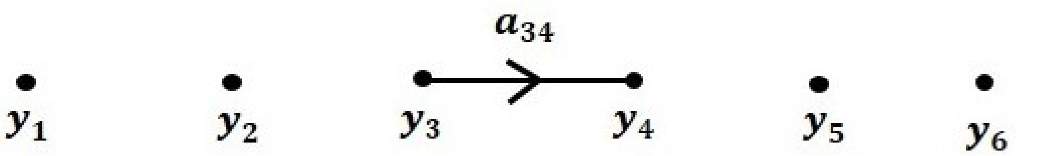
To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below −

Step 1 − Signal flow graph for $y_2 = a_{13}y_1 + a_{42}y_4$ is shown in the following figure.

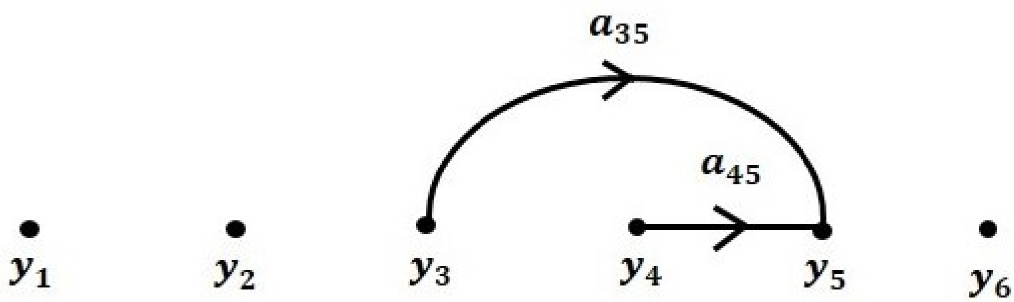


Step 2 – Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.

Step 3 – Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

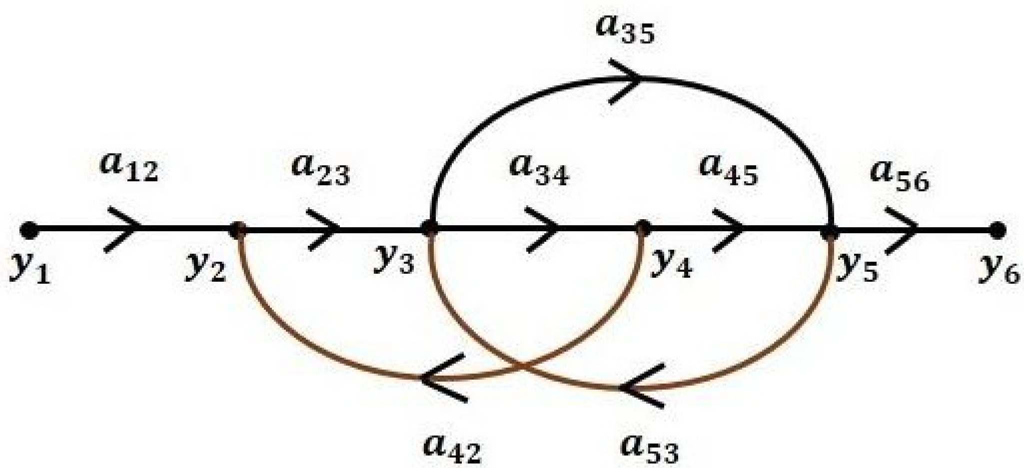


Step 4 – Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 – Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.

Step 6 – Signal flow graph of overall system is shown in the following figure.



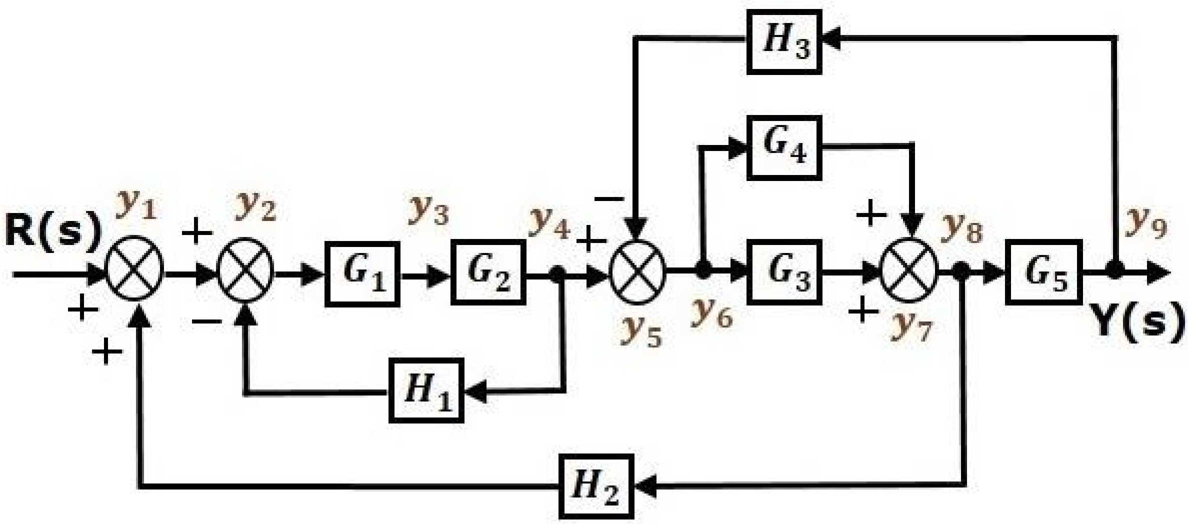
ConversionofBlockDiagramsintoSignalFlowGraphs

Followthesestepsforconvertingablock diagramintoitsequivalentsignalflowgraph.

1. Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
2. Representtheblocksofblockdiagramas**branches**insignalflowgraph.
3. Represent the transfer functions inside the blocks of block diagram as gainsof the branches in signal flow graph.
4. Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one.**For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Example

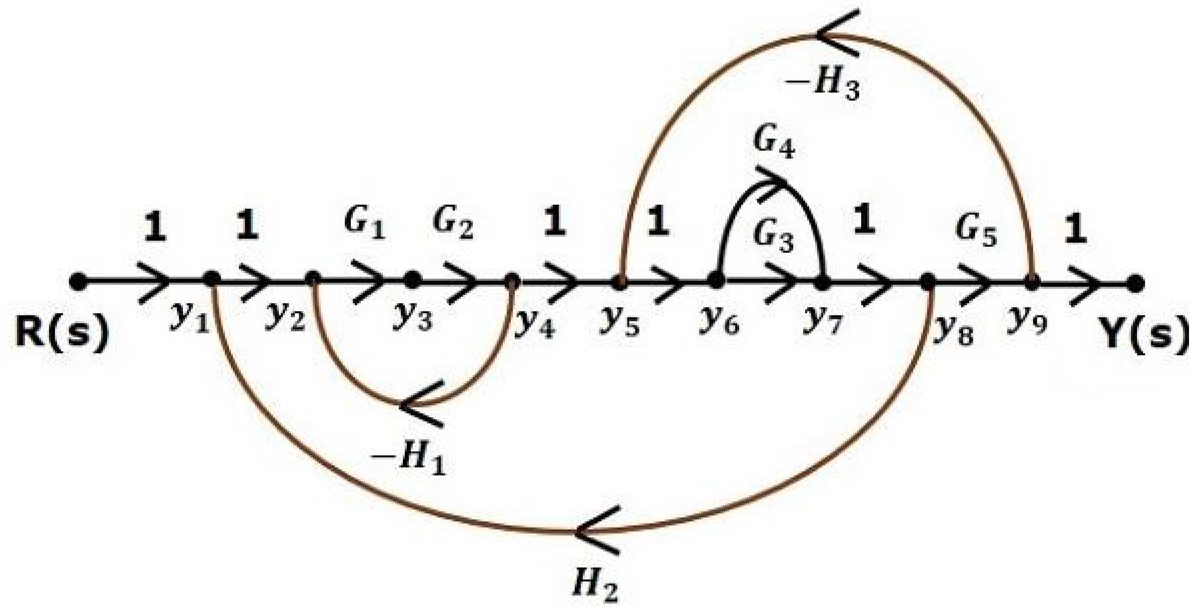
Letusconvertthefollowingblock diagramintoitsequivalentsignal flowgraph.



Representtheinputsignal R(s)andoutputsignalC(s)ofblockdiagramasinput node R(s) and output node C(s) of signal flow graph.

Justforreference,theremainingnodes($y_1$toy$_9$)arelabeledintheblockdiagram.Thereare nine nodes other than input andoutput nodes. That is fournodes forfoursumming points, fournodesforfourtake-offpointsandone nodeforthevariable betweenblocks$G_1$and G_2.

Thefollowingfigureshowstheequivalentsignalflowgraph.



LetusnowdiscusstheMason’sGainFormula.Supposethereare‘N’forwardpathsinasignal flowgraph.Thegainbetweentheinputandtheoutputnodesofasignalflowgraphis nothing but the **transfer function**of the system. It can be calculated by using Mason's gain formula.

Mason'sgainformulais

$$T=\frac{C(s)}{R(s)}=\frac{\Sigma_{i=1}^{N}P_i\Delta_i}{\Delta}$$

Where,

5. **C(s)**istheoutputnode
6. **R(s)**istheinputnode
7. **T**isthetransferfunctionorgainbetweenR(s)andC(s)

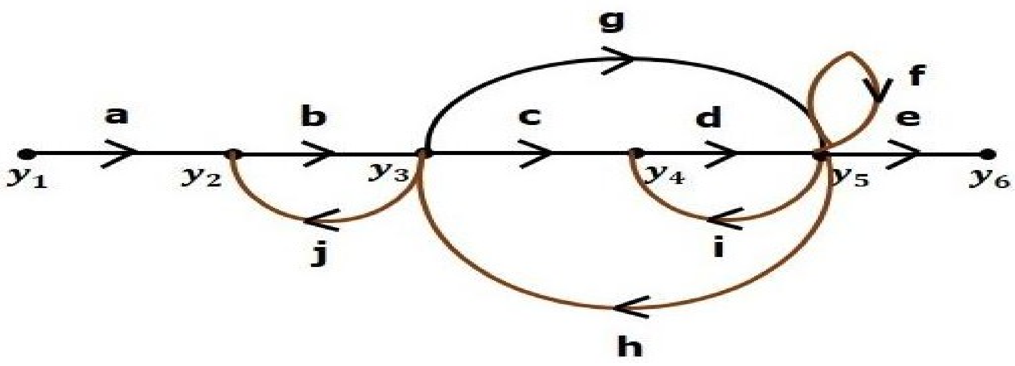
8. $P_i$istheithforwardpath gain

   Δ=1−(sumofallindividualloopgains)+(sumofgainproductsofallpossibletwo nontouchingloops)−(sumofgainproductsofallpossiblethreenontouchingloops)

   +....

   $\Delta_i$isobtainedfromΔ byremovingtheloopswhicharetouchingtheithforward path.

Considerthe followingsignalflowgraphinordertounderstandthebasicterminology involved here.


g
f
a
b
c
d
e
y_1
y_2
y_3
y_4
y_5
y_6
j
i
h

Path

It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples – $y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$ and $y_5 \rightarrow y_3 \rightarrow y_2$

Forward Path

The path that exists from the input node to the output node is known as **forward path**.

Examples – $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples – $abcde$ is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and abge is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Loop

Thepaththatstartsfromonenodeandendsatthesamenodeisknownasa **loop**.Hence,it is a closed path.

Examples – $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_3 \rightarrow y_5 \rightarrow y_3$.

Loop Gain

It is obtained by calculating the product of all branch gains of a loop.

Examples – b_j is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$ and g_h is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$.

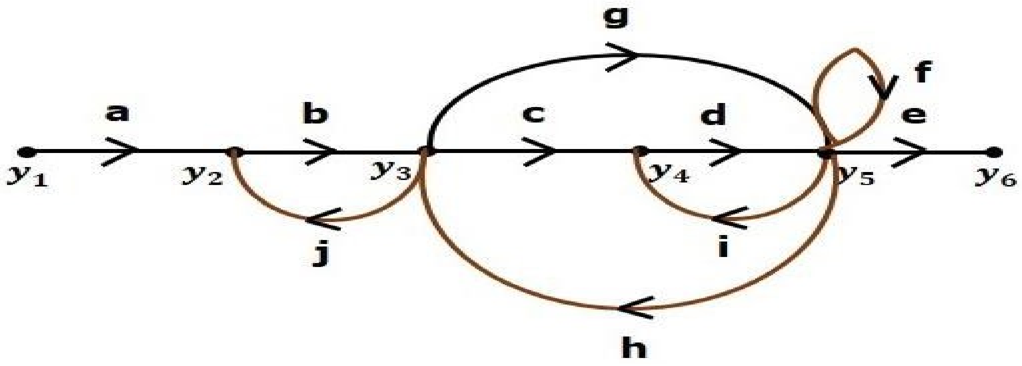
Non-touching Loops

These are the loops, which should not have any common node.

Examples – The loops, $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_4 \rightarrow y_5 \rightarrow y_4$ are non-touching.

CalculationofTransferFunctionusingMason'sGainFormula

Letusconsiderthesamesignalflowgraphforfindingtransferfunction.



9. Numberofforwardpaths,N=2.

10. Firstforwardpathis-$y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.

11. Firstforwardpathgain,p_1=abcde

12. Secondforwardpathis-$y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$

13. Secondforwardpathgain,p_2=abge

14. Numberofindividualloops,L=5.

Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

15. Numberoftwonon-touchingloops=2.

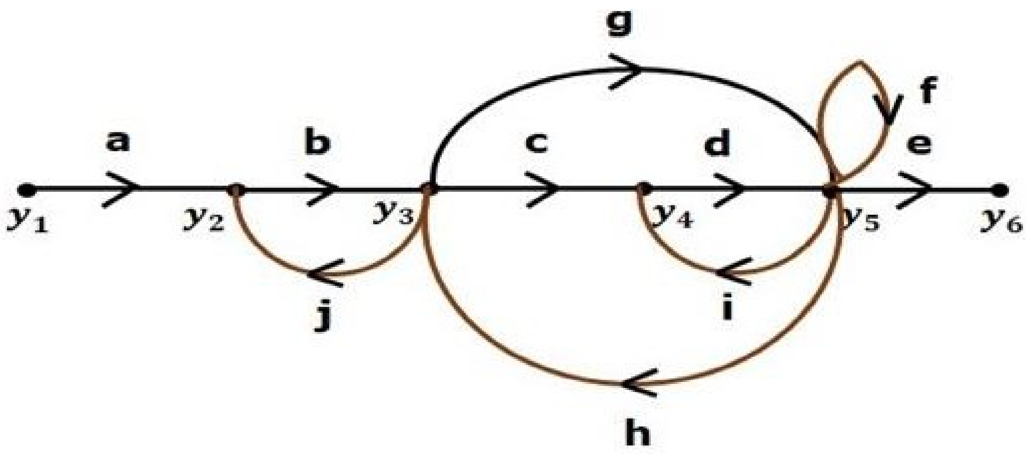
16. Firstnon-touchingloopspairis-$y_2 \rightarrow y_3 \rightarrow y_2$,$y_4 \rightarrow y_5 \rightarrow y_4$.

17. Gainproductoffirstnon-touchingloopspair$l_1 l_4$=bjdi

18. Secondnon-touchingloopspairis-$y_2 \rightarrow y_3 \rightarrow y_2$,$y_5 \rightarrow y_5$.

19. Gainproductofsecondnon-touchingloopspairis$l_1 l_5$=bjf

Higher number of (more than two) non-touching loops are not present in this signal flowgraph.We know,



- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- First forward path gain, $p_1 = abcde$.
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.
- Second forward path gain, $p_2 = abge$.
- Number of individual loops, L = 5.

- Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

  - Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.
  - Number of two non-touching loops = 2.
  - First non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_4 \rightarrow y_5 \rightarrow y_4$.
  - Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$
  - Second non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_5 \rightarrow y_5$.
  - Gain product of second non-touching loops pair is - $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (sum\,of\,all\,individual\,loop\,gains)$$
$$+(sum\,of\,gain\,products\,of\,all\,possible\,two\,nontouching\,loops)$$
$$-(sum\,of\,gain\,products\,of\,all\,possible\,three\,nontouching\,loops)+\ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta_1 = 1$.

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, N = 2 in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{2} P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

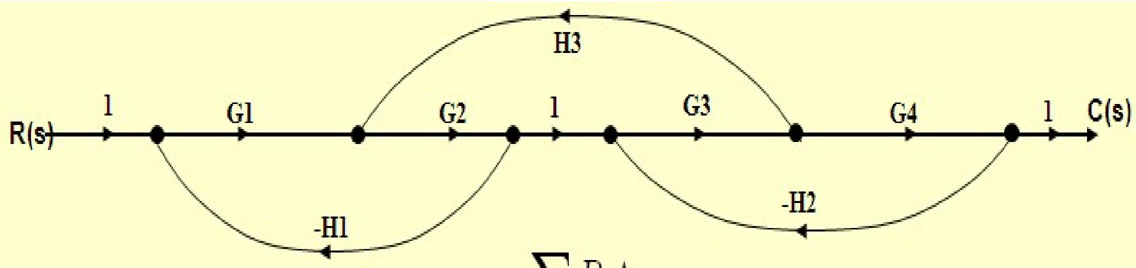
$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is -

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

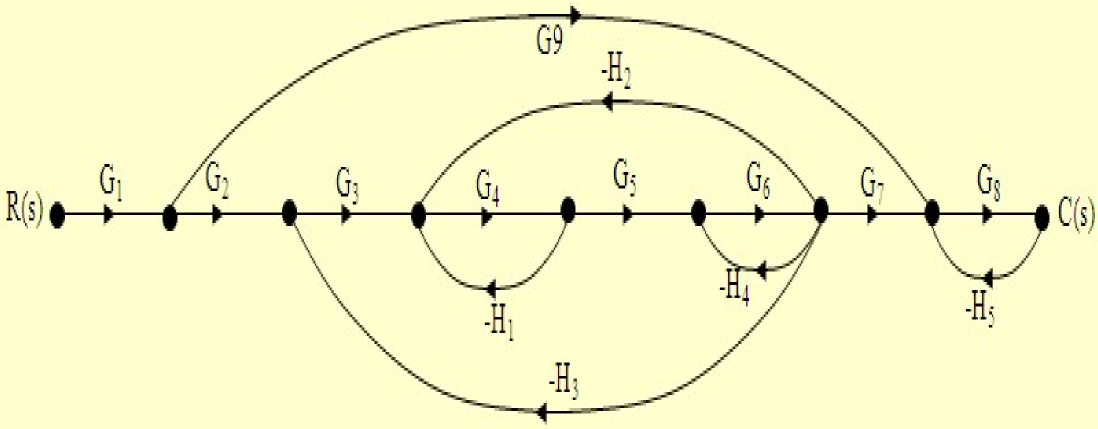
Example-1:DeterminethetransferfunctionC(s)/R(s).



$$T(s)=\frac{\sum P_k\Delta_k}{\Delta}$$

- $P_1 = G_1G_2G_3G_4 \qquad \Delta_1 = 1$ There is no P_2 or Δ_2 or more.
- $\sum L_1 = -G_1G_2H_1 + G_2G_3H_3 - G_3G_4H_2$
- $\sum L_2 = G_1G_2G_3G_4H_1H_2$
- $\Delta = 1 - \sum L_1 + \sum L_2 = 1 + G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2$
- $T(s) = \dfrac{\sum P_1\Delta_1}{\Delta} = \dfrac{G_1G_2G_3G_4}{1+G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2}$

Example-2:DeterminethetransferfunctionC(s)/R(s).



$$M_1 = G_1G_2G_3G_4G_5G_6G_7G_8 \qquad \Delta_1 = 1$$

$$M_2 = G_1G_9G_8 \qquad \Delta_2 = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2] + G_4H_1G_6H_4$$

$$= 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4$$

$$\Delta = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2 - G_8H_5]$$

$$+\left[G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3\right]$$

$$+G_4H_1G_6H_4G_8H_5$$

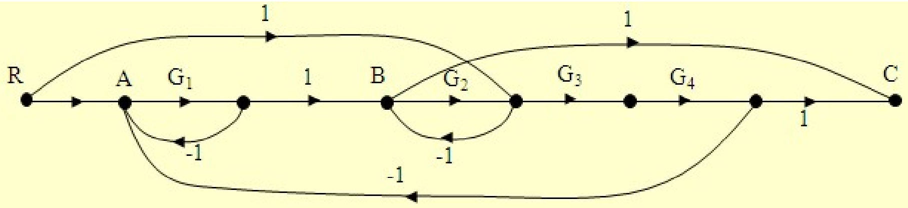
$$\Delta = 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_8H_5$$

$$+G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3$$

$$+G_4H_1G_6H_4G_8H_5$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5G_6G_7G_8 + G_1G_9G_8[1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4]}{\Delta}$$

Example-3:DeterminethetransferfunctionC(s)/R(s).



$$\mathbf{M_1} = G_1G_2G_3G_4 \qquad \Delta_1 = 1$$

$$\mathbf{M_2} = G_1 \qquad \Delta_2 = 1$$

$$\mathbf{M_3} = G_3G_4 \qquad \Delta_3 = 1 + G_1$$

$$\mathbf{M_4} = -1 \qquad \Delta_4 = 1 + G_1$$

$$\mathbf{M_5} = -G_3G_4G_1 \qquad \Delta_5 = 1$$

$$\Delta = 1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2 = 1 + G_1 + G_2 + G_1G_2G_3G_4 + G_1G_2$$

$$\text{a)}\quad \frac{C}{M} = \frac{G_1G_2G_3G_4 + G_1 + G_3G_4(1+G_1) - (1)(1+G_1) - G_1G_3G_4}{1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2}$$

UNIT-II

TIMERESPONSEANALYSIS

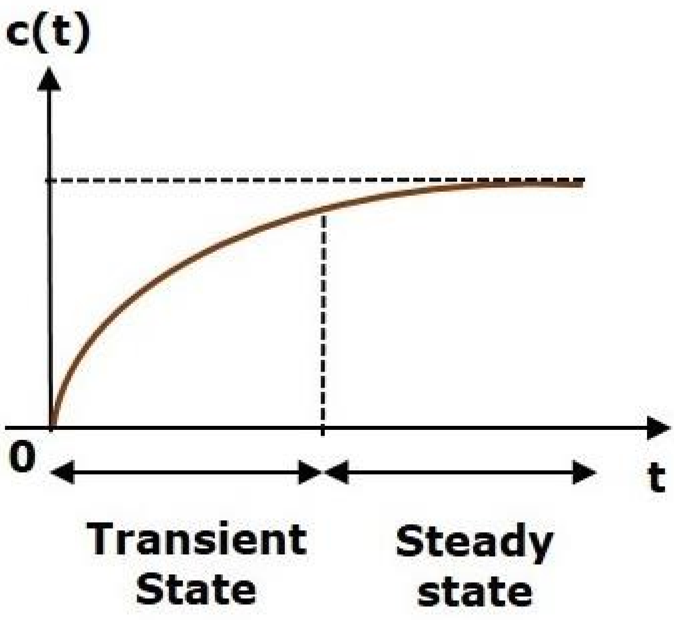
We can analyze the response of the control systems in both the time domain and the frequency domain. We will discuss frequency response analysis of control systems in later chapters. Let us now discuss about the time response analysis of control systems.

WhatisTimeResponse?

Iftheoutputofcontrolsystemforaninputvarieswithrespecttotime,thenitiscalled the **time response** of the control system. The time response consists of two parts.

1. Transientresponse
2. Steady state response

Theresponseofcontrolsystemintime domainisshowninthefollowing figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

3. $c_{tr}(t)$isthetransientresponse

4. $c_{ss}(t)$isthesteadystate response

TransientResponse

Afterapplying input tothe control system,output takes certain time to reach steady state. So,the outputwillbeintransientstatetillitgoes toasteadystate.Therefore,the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times constant.

Mathematically,wecanwriteitas

$$\lim_{t\to\infty} c_{tr}(t) = 0$$

SteadystateResponse

Thepartofthetimeresponsethatremainsevenafterthetransientresponsehaszerovalue forlargevaluesof‘t’isknownas **steadystateresponse**.Thismeans,thetransientresponse will be zero even during the steady state.

Example

Letusfindthetransientandsteadystatetermsofthetimeresponseofthe controlsystem

$$c(t) = 10 + 5e^{-t}$$

Here, the second term $5e^{-t}$ willbezeroas**t**denotesinfinity.So,thisisthe**transientterm**. Andthefirstterm10remainsevenas **t**approachesinfinity.So,thisisthe **steadystateterm**.
StandardTestSignals

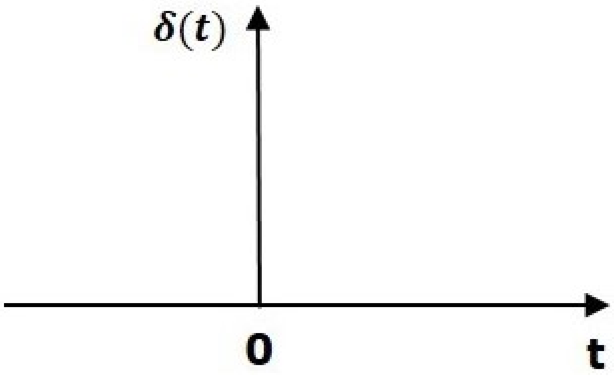
The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

UnitImpulseSignal

Aunit impulsesignal,δ(t)isdefinedas

$$\delta(t) = 0 \text{ for } t \neq 0$$
$$\text{and } \int_{0^-}^{0^+} \delta(t)dt = 1$$

The following figure shows unit impulse signal.



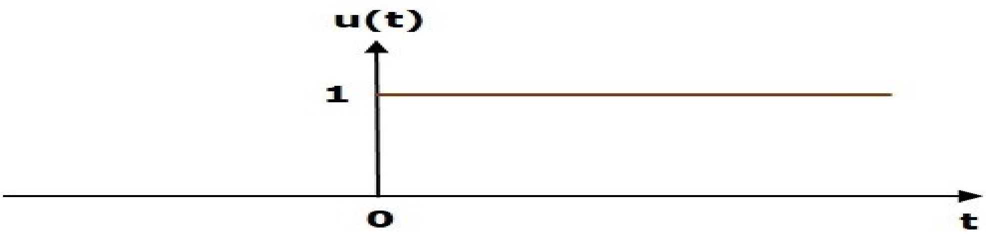
So,theunitimpulsesignalexistsonlyat‘t’isequaltozero.Theareaofthissignalundersmall interval oftime around‘t’ is equalto zero is one. The value of unit impulse signal is zerofor all other values of‘t’.

UnitStep Signal

Aunitstepsignal,u(t)isdefinedas

$$u(t) = 1; t \geq 0$$
$$= 0; t < 0$$

Followingfigureshowsunitstepsignal.



So, the unit step signal exists for all positive values of‘t’ including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of‘t’.

UnitRampSignal

Aunitrampsignal, r(t)isdefined as

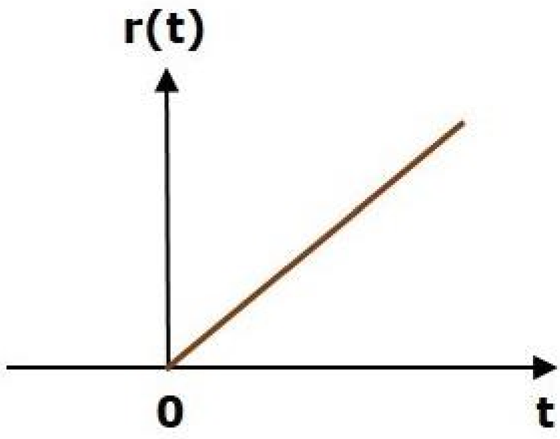
$$r(t) = t; t \geq 0$$

$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of't' including zero. And its value increaseslinearlywithrespect to't'duringthis interval.Thevalueofunit rampsignaliszero for all negative values of't'.

UnitParabolicSignal

Aunit parabolicsignal,p(t)isdefinedas,

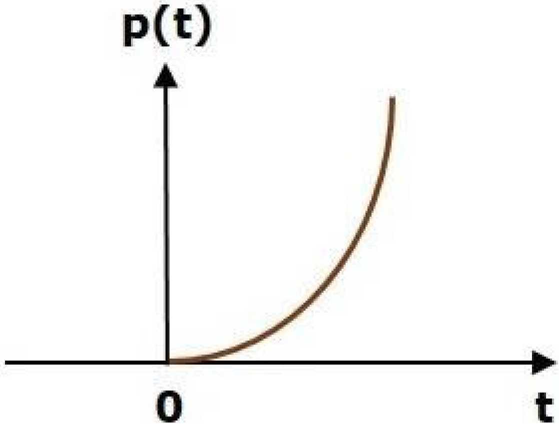
$$p(t) = \frac{t^2}{2}; t \geq 0$$

$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

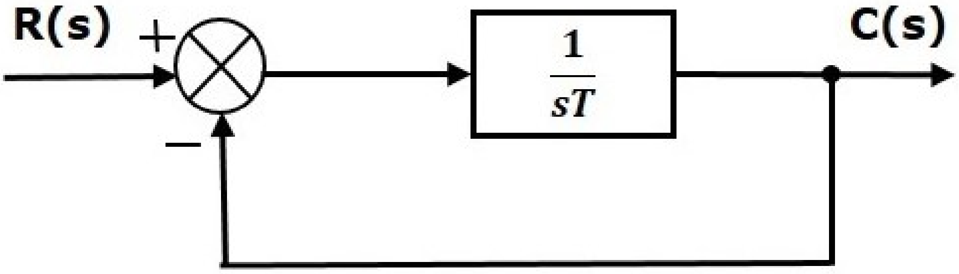
$$p(t) = \frac{t^2}{2} u(t)$$

The following figure shows the unit parabolic signal.



So,the unitparabolicsignalexistsforallthe positivevalues of‘t’includingzero.Anditsvalue increases non-linearly with respect to‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of‘t’.

In this chapter, let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

ImpulseResponseofFirstOrderSystem

Considerthe**unitimpulsesignal**asan inputtothefirstordersystem.

So,r(t)=δ(t)

ApplyLaplacetransformonboththesides.

R(s) =1

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Rearrangetheaboveequationinoneofthestandardformsoflaplace transforms.

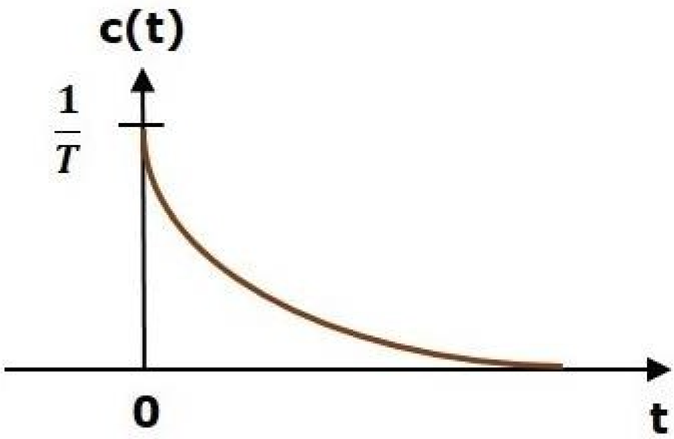
$$C(s) = \frac{1}{T\left(s+\frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s+\frac{1}{T}}\right)$$

ApplyingInverseLaplaceTransformonboththesides,

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

Theunitimpulseresponseisshowninthefollowing figure.



The**unitimpulseresponse**,c(t)isanexponentialdecayingsignalforpositivevaluesof‘t’and it is zero for negative values of ‘t’.

StepResponseofFirstOrderSystem

Considerthe**unitstep signal**asaninputtofirstordersystem. So,

$$r(t)=u(t)$$

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s\,(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\,(sT+1)} = \frac{A\,(sT+1) + Bs}{s\,(sT+1)}$$

On boththesides,thedenominatortermisthesame.So, theywillgetcancelledbyeach other. Hence, equate the numerator terms.

1=A(sT+1)+Bs

By equating the constant terms on both the sides, you will get A = 1.

Substitute,A=1andequatethecoefficient ofthe **s**termsonboththesides.

0=T+B

⇒B=−T

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Substitute,A=1andB=−TinpartialfractionexpansionofC(s)

ApplyinverseLaplacetransformonboththesides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The**unitstepresponse**,c(t)hasboththetransientandthesteadystate terms.

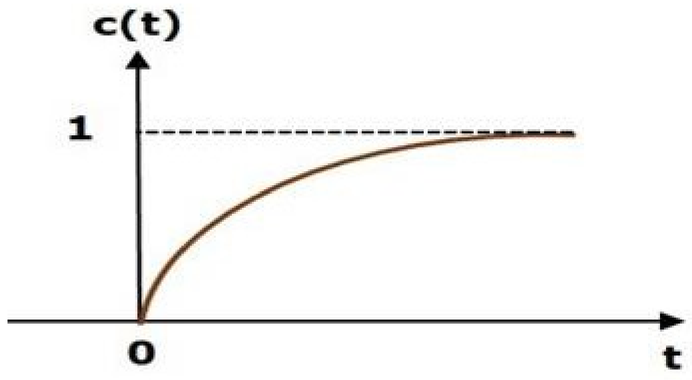
Thetransientterm intheunitstepresponse is-

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)}u(t)$$

Thesteadystatetermintheunitstepresponseis − The

$$c_{ss}(t) = u(t)$$

following figure shows the unit step response



The value of the**unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is graduallyincreasingfromzerovalueandfinallyreachestooneinsteadystate.So,thesteady state value depends on the magnitude of the input.

RampResponseofFirstOrderSystem

Considerthe**unitrampsignal**asaninputtothefirstorder system.

So,r(t)=tu(t)

ApplyLaplacetransformonboththesides.

$$R(s)=\frac{1}{s^2}$$

Consider the equation, $C(s)=\left(\frac{1}{sT+1}\right)R(s)$

Substitute, $R(s)=\frac{1}{s^2}$ in the above equation.

$$C(s)=\left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right)=\frac{1}{s^2(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelledby eachother. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute,A=1andequatethecoefficient ofthe stermsonboththesides.

$$0=T+B \Rightarrow B=-T$$

Similarly,substituteB=−Tandequatethe coefficientofs^{2}termsonboththesides.Youwill get C=T^2

SubstituteA=1,B=−TandC=T^2inthepartialfractionexpansionof C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

ApplyinverseLaplacetransformonboththesides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The**unitrampresponse**,c(t) hasboththe transientandthe steadystateterms. The

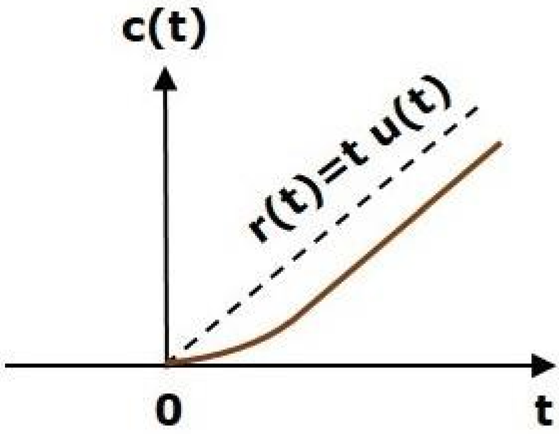
$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

transient term in the unit ramp response is

Thesteadystatetermintheunitrampresponse is–

$c_{ss}(t) = (t - T)u(t)$

Thefigurebelowistheunit rampresponse:



The**unitrampresponse**,c(t)followstheunit rampinputsignalforallpositive valuesoft. But, there is a deviation of T units from the input signal.

ParabolicResponseofFirstOrderSystem

Considerthe**unitparabolicsignal**asaninputtothefirstordersystem.

So, $r(t) = \frac{t^2}{2}u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T,\ T^2\ and\ -T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

ApplyinverseLaplacetransformonboththesides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The**unitparabolicresponse**,c(t) hasboththetransientandthesteadystateterms. The transient term in the unit parabolic response is

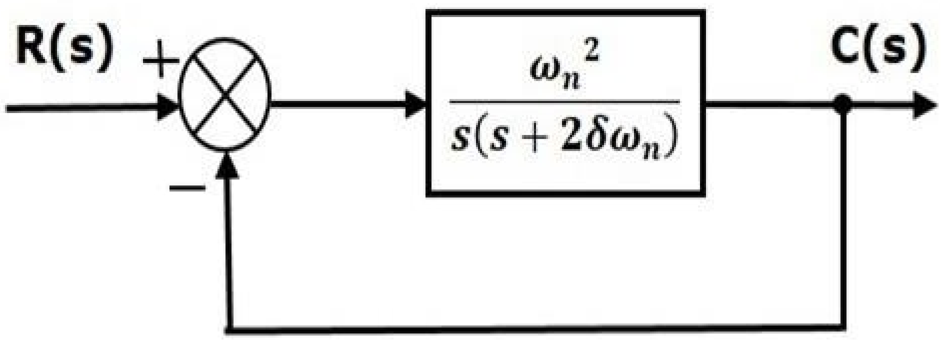
$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

Thesteadystatetermintheunitparabolicresponseis

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

From these responses, we can conclude that the first order control systems are not stable withtherampandparabolicinputsbecausetheseresponsesgoonincreasingevenatinfinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn’t have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

In this chapter, let us discuss the time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop transferfunction, $\omega_n^2 / s(s+2\delta\omega n)$is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

Thepowerof‘s’istwointhedenominatorterm.Hence,theabovetransferfunctionisofthe second order and the system is said to be the **second order system**.

Thecharacteristicequationis-

$$s^2+2\delta\omega_n s+\omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2-4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2-1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2-1}$$

b

1. Thetworootsareimaginarywhen δ=0.
2. Thetworootsarerealandequalwhenδ=1.
3. Thetworootsarerealbutnotequalwhenδ >1.

4. Thetworootsarecomplexconjugatewhen0<δ<1.

We can write C(s) equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$

Where,

5. **C(s)**istheLaplacetransformoftheoutputsignal, c(t)
6. **R(s)**istheLaplacetransformoftheinputsignal, r(t)
7. **ω_n**isthenaturalfrequency
8. **δ**isthedamping ratio.

Followthesestepstogettheresponse(output)ofthesecondordersysteminthetime domain.

Take Laplace transform of the input signal, $r(t)$.

Consider the equation, $C(s) = \left(\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2} \right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$.

StepResponseofSecondOrderSystem

Consider the unit step signal as an input to the second order system.Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\,and\ -\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\, and\ -2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2 + \omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t}\sin(\omega_d t)\right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right) u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right) u(t)$$

So,theunitstepresponseofthesecondordersystemishavingdampedoscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case4:δ>1

Wecanmodifythedenominatortermofthetransferfunctionasfollows−

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1})(s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1})}$$

$$= \frac{A}{s} + \frac{B}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}} + \frac{C}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$

and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\left(\frac{1}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}}\right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)\left(\frac{1}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right)u(t)$$

Since it is over damped, the unit step response of the second order system when δ > 1 will never reach step input in the steady state.

ImpulseResponseofSecondOrderSystem

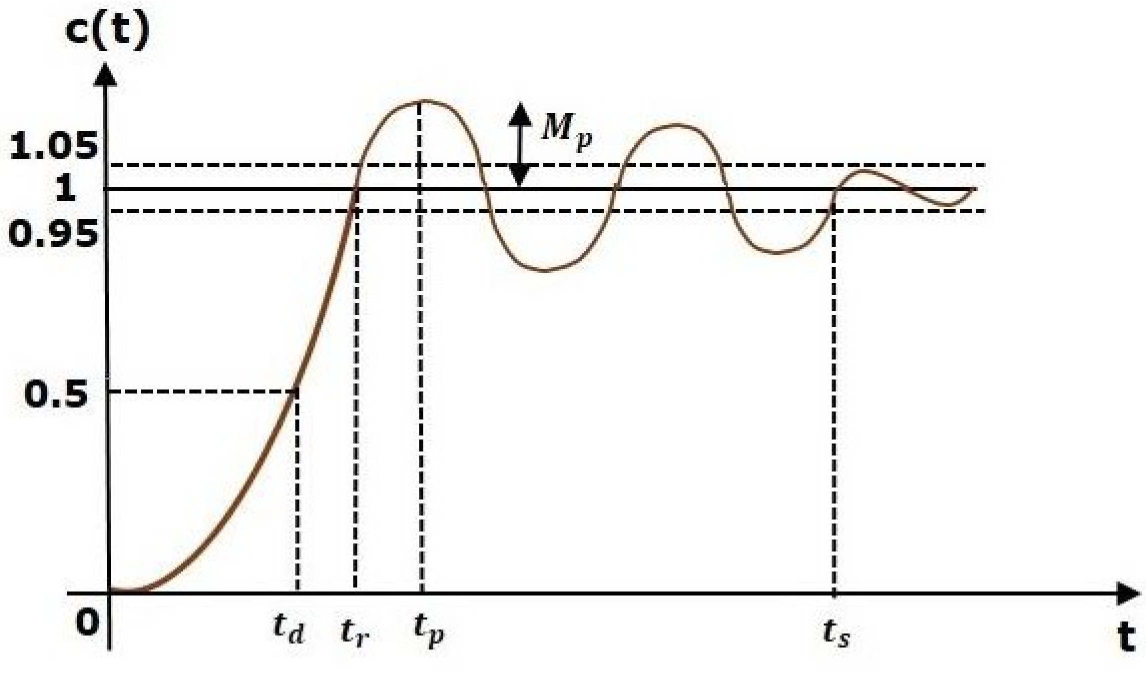
The**impulseresponse** ofthesecondordersystemcanbeobtainedbyusinganyoneofthese two methods.

1. Followtheprocedureinvolvedwhilederivingstepresponsebyconsideringthevalue of R(s) as 1 instead of 1/s.
2. Dothedifferentiationofthestepresponse.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	Impulse response for t ≥ 0
$\delta = 0$	$\omega_n \sin(\omega_n t)$
$\delta = 1$	$\omega_n^2 t e^{-\omega_n t}$
$0 < \delta < 1$	$\left(\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}}\right)\left(e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t}\right)$

Inthischapter,letusdiscussthetimedomainspecificationsofthesecondordersystem.The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settlingtimeisknownastransient responseandtheresponseafterthesettlingtime isknown as steady state response.

DelayTime

Itisthetime required for the responseto reach **halfofitsfinal value** fromthe zero instant. It is denoted by tdtd.

Considerthestep responseofthesecondordersystemfort≥ 0,when‘δ’liesbetweenzero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

RiseTime

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the**under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by **t_r**.

Att=t_1=0, c(t)=0.

Weknowthatthefinalvalueofthestepresponseisone.Therefore,att=t2,thevalueofstep response is one. Substitute, these values in the following equation.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore\ t_r = \frac{\pi - \theta}{\omega_d}$$

Fromaboveequation,wecanconcludethattherisetime $t_r$andthedampedfrequency $\omega_d$are inversely proportional to each other.

PeakTime

Itisthetimerequired forthe responsetoreachthe **peakvalue**forthe firsttime.Itis denoted by t_p. At $t=t_p$the first derivate of the response is zero.

Weknowthestepresponseofsecondordersystemforunder-dampedcaseis

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\left[\omega_d\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta)\right]$$

$$\Rightarrow \omega_n\sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta)\cos(\omega_d t_p + \theta) - \cos(\theta)\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

Fromtheaboveequation,wecanconcludethatthepeaktime$t_p$andthedamped frequency $\omega_d$are inversely proportional to each other.

PeakOvershoot

Peak overshoot**M_p** is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically,wecanwriteitas

$$Mp=c(t_p)-c(\infty)$$

Where,$c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of theresponse.

At $t=t_p$, the response $c(t)$ is-

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right)\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right)(-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

Fromtheaboveequation,wecanconcludethatthepercentageofpeakovershoot %Mpwill decrease if the damping ratio δ increases.

Settlingtime

Itisthetimerequiredfortheresponsetoreachthesteadystateandstaywithinthespecified tolerance bands aroundthe finalvalue. Ingeneral,the tolerance bands are 2% and5%. The settling time is denoted by ts.

Thesettlingtimefor5%tolerancebandis–

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

Thesettlingtimefor2%tolerancebandis–

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where,τisthetimeconstantandisequalto1/$\delta\omega_n$.

1. Both the settling time ts and the time constant τare inversely proportional to the damping ratio δ.
2. Boththesettlingtimetsandthetimeconstantτareindependentofthesystemgain. Thatmeanseventhesystemgainchanges,thesettlingtime tsandtime constant τ will never change.

Example

Let us now find the time domain specifications of a control system having the closed loop transfer function when the unit step signal is applied as an input to this control system.

We know that the standard form of the transfer function of the second order closed loop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Byequatingthesetwotransferfunctions,wewillgettheun-dampednaturalfrequency $\omega_n$as 2 rad/sec and the damping ratio δ as 0.5.

Weknowtheformulafordampedfrequency $\omega_d$as

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1-(0.5)^2}$$

$$\Rightarrow \omega_d = 1.732\ rad/sec$$

Substitute, δ value in following relation

$$\theta = \cos^{-1}\delta$$

$$\Rightarrow \theta = \cos^{-1}(0.5) = \frac{\pi}{3}\ rad$$

Substitutetheabovenecessaryvaluesintheformulaofeachtimedomainspecificationand simplify in ordertoget the values oftime domain specifications forgiventransferfunction.

Thefollowingtableshowstheformulaeoftimedomainspecifications,substitutionof necessary values and the final values

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$	$t_d = \frac{1+0.7(0.5)}{2}$	t_d =0.675 sec
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$	$t_r = \frac{\pi-(\frac{\pi}{3})}{1.732}$	t_r =1.207 sec
Peak time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	t_p =1.813 sec
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}\right) \times 100\%$	$\%M_p = \left(e^{-\left(\frac{0.5\pi}{\sqrt{1-(0.5)^2}}\right)}\right) \times 100\%$	$\%\ M_p$ =16.32%
Settling time for 2% tolerance band	$t_s = \frac{4}{\delta\omega_n}$	$t_S = \frac{4}{(0.5)(2)}$	t_s =4 sec

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} E(s)$$

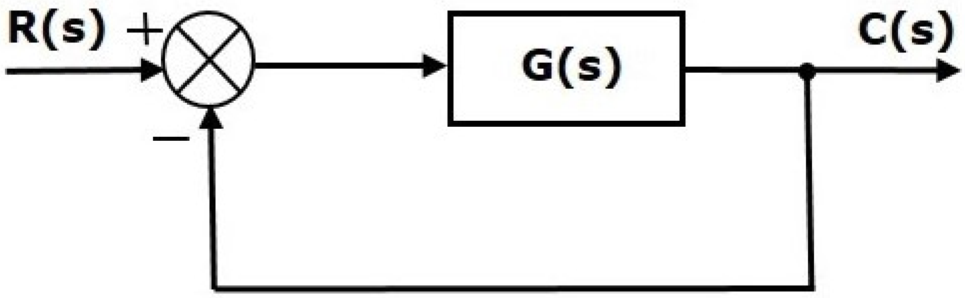
Where,

E(s)istheLaplacetransformoftheerrorsignal,e(t)

Let us discuss howto find steady state errors for unity feedbackand non-unity feedbackcontrol systems one by one.

SteadyStateErrorsforUnityFeedbackSystems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,

- R(s) is the Laplace transform of the reference Input signal $r(t)$
- C(s) is the Laplace transform of the output signal $c(t)$

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Thefollowingtableshowsthesteadystateerrorsandtheerrorconstantsforstandardinput signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \to 0} s^2G(s)$

Where,Kp,KvandKa are position error constant, velocity error constant and acceleration error constant respectively.

Note– If any of the above input signals has the amplitude other than unity, then multiply corresponding steady state error with that amplitude.

Note– We can't define the steady state error for the unit impulse signal because, it exists onlyatorigin.So,wecan'tcomparetheimpulseresponsewiththeunitimpulseinput as **t** denotes infinity

Example

Let us find the steady state error for an input signal $r(t) = \left(5 + 2t + \frac{t^2}{2}\right)u(t)$ of unity negative feedback control system with $G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s\to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s\to 0} sG(s)$ $= \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s\to 0} s^2G(s)$ $= 1$	$e_{ss3} = \frac{1}{k_a} = 1$

Wewillgettheoverallsteadystateerror,byaddingtheabovethreesteadystate errors.

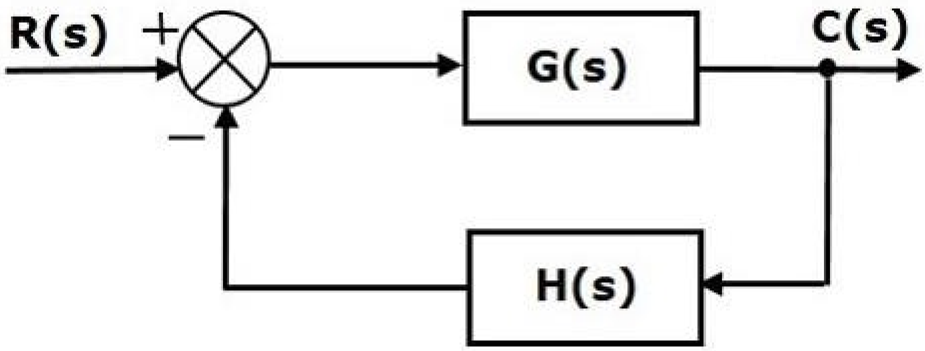
$$e_{ss}= e_{ss1}+e_{ss2}+e_{ss3}$$

$$\Rightarrow e_{ss}=0+0+1=1 \Rightarrow e_{ss}=0+0+1=1$$

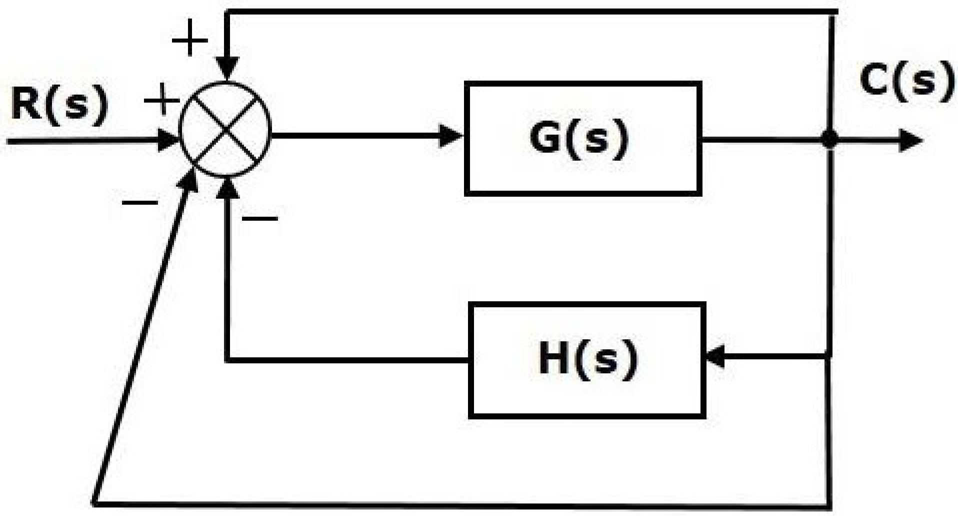
Therefore,wegotthesteadystateerrore_{ss}as**1**forthisexample.

SteadyStateErrorsforNon-Unity FeedbackSystems

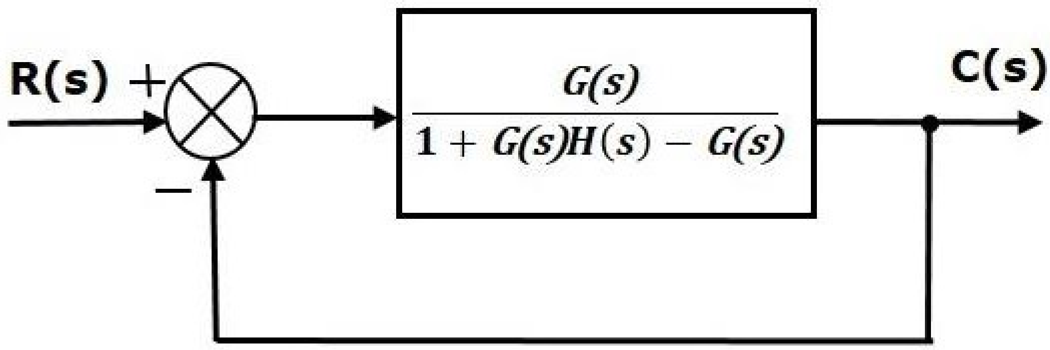
Considerthefollowingblockdiagramofclosedloopcontrolsystem,whichishavingnonunity negative feedback.



We can find the steady state errors only for the unity feedback systems. So, we have to convert the non-unity feedback system into unity feedback system. For this, include one unity positive feedback path and one unity negative feedback path in the above block diagram. The new block diagram looks like as shown below.



Simplify the above block diagram by keeping the unity negative feedback as it is. The following is the simplified block diagram


R(s)
+
−
G(s)
1 + G(s)H(s) − G(s)
C(s)

This block diagram resembles the block diagram ofthe unity negative feedback closed loop control system. Here, the single block is having the transfer function G(s) / [1+G(s)H(s)−G(s)]instead ofG(s).You can now calculate the steady state errors by using steady state error formula given for the unity negative feedback systems.

Note−Itismeaninglesstofindthe steadystateerrorsforunstableclosedloopsystems.So, wehavetocalculatethesteadystateerrors only forclosedloopstablesystems.Thismeans weneedtocheckwhetherthecontrolsystemisstableornotbeforefindingthesteadystate errors. In the next chapter, we will discuss the concepts-related stability.

Thevarioustypesofcontrollersareusedtoimprovetheperformanceofcontrolsystems.In thischapter,wewilldiscussthebasiccontrollerssuchastheproportional,thederivativeand the integral controllers.

ProportionalController

Theproportionalcontrollerproducesanoutput,whichisproportionaltoerrorsignal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

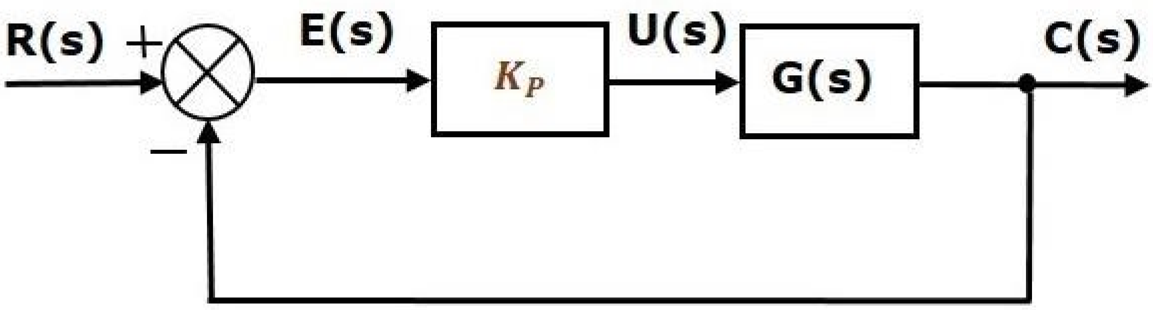
Therefore,thetransferfunctionoftheproportionalcontrolleris KPKP. Where,

U(s)istheLaplacetransformoftheactuatingsignalu(t) E(s) is

the Laplace transform of the error signal e(t)

$K_P$istheproportionalityconstant

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalongwiththe proportional controller is shown in the following figure.



DerivativeController

Thederivativecontrollerproducesan output,whichisderivativeoftheerrorsignal.

$$u(t) = K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

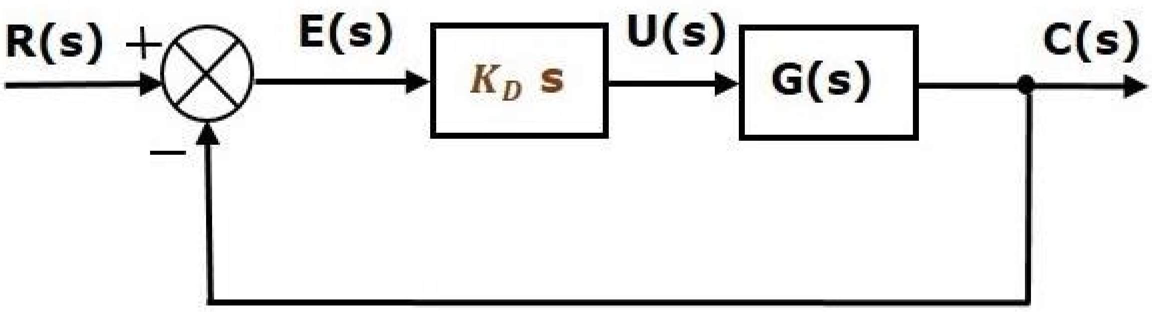
Apply Laplace transform on both sides.

$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore,thetransferfunctionofthederivativecontrolleris $K_D s$.
Where, KD is the derivative constant.

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalong withthe derivative controller is shown in the following figure.



Thederivativecontrollerisusedtomaketheunstablecontrolsystemintoastableone.

IntegralController

Theintegralcontrollerproducesanoutput,whichisintegraloftheerrorsignal.

$$u(t) = K_I \int e(t)dt$$

Apply Laplace transform on both the sides -

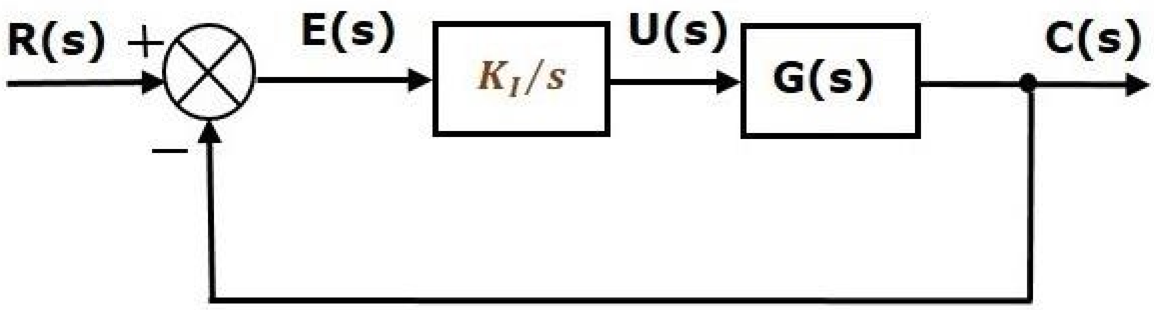
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where,KIKIistheintegral constant.

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalongwiththe integral controller is shown in the following figure.



Theintegralcontrollerisusedtodecreasethesteadystate error. Let us

now discuss about the combination of basic controllers.

ProportionalDerivative(PD)Controller

The proportionalderivative controllerproduces anoutput, which is the combination ofthe outputs of proportional and derivative controllers.

$$u(t) = K_P e(t) + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

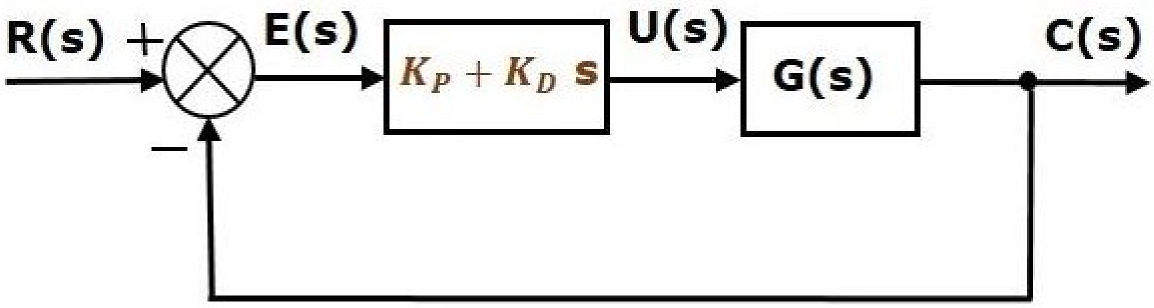
Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore,thetransferfunctionoftheproportionalderivativecontrolleris$K_P+K_D s$.

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalongwiththe proportional derivative controller is shown in the following figure.



Theproportionalderivativecontrollerisusedtoimprovethestabilityofcontrolsystem without affecting the steady state error.

ProportionalIntegral(PI)Controller

Theproportionalintegralcontrollerproducesanoutput,whichisthecombinationofoutputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt$$

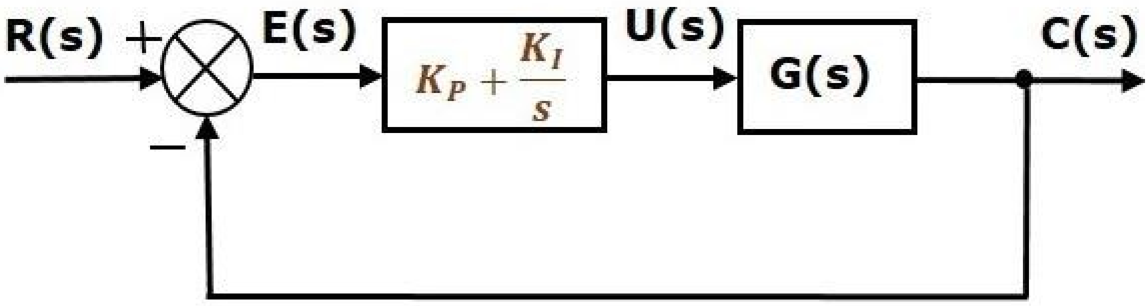
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalongwiththe proportional integral controller is shown in the following figure.



Theproportionalintegralcontrollerisusedtodecreasethesteadystateerrorwithout affecting the stability of the control system.

ProportionalIntegralDerivative(PID)Controller

Theproportionalintegralderivativecontrollerproducesanoutput,whichisthecombination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

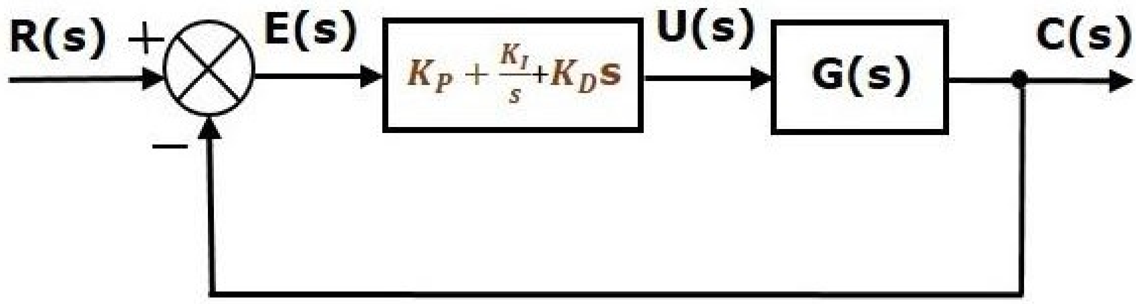
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

Theblockdiagramofthe unitynegativefeedback closedloopcontrolsystemalongwiththe proportional integral derivative controller is shown in the following figure.


R(s)
+
−
E(s)
$K_P + \frac{K_I}{s} + K_D s$
U(s)
G(s)
C(s)

UNIT-III

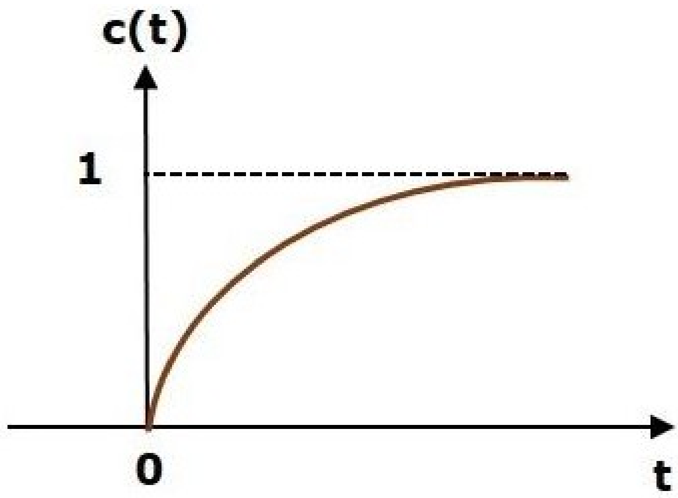
STABILITYANALYSISINS-DOMAIN

Stabilityisanimportantconcept.Inthischapter,letusdiscussthestabilityofsystemand types of systems based on stability.

WhatisStability?

Asystemissaidtobestable,ifitsoutputisundercontrol.Otherwise,itissaidtobeunstable. A **stable system** produces a bounded output for a given bounded input.

Thefollowingfigureshowstheresponseofastablesystem.



This is the response of first order control system for unit step input. This response has the valuesbetween0and 1.So,it isboundedoutput. Weknowthattheunitstepsignalhasthe valueofoneforallpositivevaluesof **t**includingzero.So,itisboundedinput.Therefore,the first order control system is stable since both the input and the output are bounded.

TypesofSystemsbasedonStability

Wecanclassifythesystemsbasedonstabilityas follows.

1. Absolutelystablesystem
2. Conditionallystablesystem
3. Marginallystablesystem

AbsolutelyStableSystem

If the system is stable for all therange ofsystemcomponent values, thenit isknownas the**absolutely stable system**. The open loop control system is absolutely stable if all the polesoftheopen looptransferfunctionpresentinlefthalfof **‘s’plane**.Similarly,theclosed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the ‘s’ plane.

ConditionallyStableSystem

Ifthesystemisstableforacertainrangeofsystemcomponentvalues,thenitisknown as **conditionally stable system**.

MarginallyStableSystem

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. Theopen loopcontrolsystemismarginallystable ifanytwopolesofthe open looptransfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginaryaxis.Inthischapter,letusdiscussthestabilityanalysisinthe **'s'**domainusingthe Routh-Hurwitz stability criterion. In this criterion, we require the characteristic equation to find the stability of the closed loop control systems.

Routh-HurwitzStabilityCriterion

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient conditionforstability.Ifanycontrolsystemdoesn'tsatisfythenecessarycondition,thenwe cansaythatthecontrolsystemisunstable.But,ifthecontrolsystemsatisfiesthenecessary condition,thenitmayormaynotbestable.So,thesufficientconditionishelpfulforknowing whether the control system is stable or not.

NecessaryConditionforRouth-HurwitzStability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Considerthecharacteristicequationofthe order'n'is-

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s^1 + a_ns^0 = 0$$

Notethat,thereshouldnotbeanytermmissinginthe**n**$^{\mathbf{th}}$ordercharacteristicequation.This means that the**n**$^{\mathbf{th}}$order characteristic equation should not have any coefficient that is of zero value.

SufficientConditionforRouth-HurwitzStability

ThesufficientconditionisthatalltheelementsofthefirstcolumnoftheRoutharrayshould have the same sign. This means that allthe elements of the first column of the Routh array should be either positive or negative.

RouthArrayMethod

If all the roots of the characteristic equation exist to the left half of the ‘s’ plane, then the controlsystem isstable. Ifat least one root ofthe characteristicequationexists tothe right halfofthe ‘s’plane,thenthecontrolsystemisunstable.So,wehavetofindtherootsofthe characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So,toovercomethisproblemtherewehavethe **Routharraymethod**.Inthismethod,there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the ‘s’ plane and the control system is unstable.

FollowthisprocedureforformingtheRouthtable.

4. Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of snand continue up to the coefficient of s0.

5. FilltheremainingrowsoftheRoutharraywiththeelementsasmentionedinthetable below. Continue this process till you get the first column element of**row**s0s0isan. Here, an is the coefficient of s0 in the characteristic polynomial.

Note− If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

ThefollowingtableshowstheRoutharray ofthenthordercharacteristic polynomial.

$$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\ldots+a_{n-1}s^1+a_ns^0$$

s^n	a_0	a_2	a_4	a_6	$\ldots$	$\ldots$
s^{n-1}	a_1	a_3	a_5	a_7	$\ldots$	$\ldots$
s^{n-2}	$b_1=\frac{a_1a_2-a_3a_0}{a_1}$	$b_2=\frac{a_1a_4-a_5a_0}{a_1}$	$b_3=\frac{a_1a_6-a_7a_0}{a_1}$	$\ldots$	$\ldots$	$\ldots$
s^{n-3}	$c_1=\frac{b_1a_3-b_2a_1}{b_1}$	$c_2=\frac{b_1a_55-b_3a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

Example:

Letusfindthestabilityofthecontrolsystemhavingcharacteristic equation,

$$s^4+3s^3+3s^2+2s+1=0$$

Step1−VerifythenecessaryconditionfortheRouth-Hurwitzstability. All the coefficients of the characteristic polynomial, $s^4+3s^3+3s^2+2s+1$ arepositive.So,thecontrolsystemsatisfiesthenecessary condition.

Step2−FormtheRoutharrayforthegivencharacteristicpolynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3\times3)-(2\times1)}{3}=\frac{7}{3}$	$\frac{(3\times1)-(0\times1)}{3}=\frac{3}{3}$ $=1$	
s^1	$\frac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}}$ $=\frac{5}{7}$		
s^0	1		

Step3–Verifythesufficientconditionforthe Routh-Hurwitzstability.

Allthe elements of the first columnof the Routh array are positive. There is nosignchange in the first column of the Routh array. So, the control system is stable.

SpecialCasesofRouthArray

Wemaycomeacrosstwotypesofsituations,whileforming theRouthtable.It isdifficultto complete the Routh table from these two situations.

Thetwospecialcasesare−

6. Thefirstelementofanyrowof theRouth'sarrayiszero.
7. AlltheelementsofanyrowoftheRouth'sarrayare zero.

Letusnowdiscusshowtoovercomethedifficultyinthesetwocases,onebyone.

FirstElement ofanyrowoftheRouth'sarrayiszero

IfanyrowoftheRouth'sarraycontainsonlythefirstelementaszeroandatleastoneofthe remainingelementshavenon-zerovalue,thenreplacethefirstelementwithasmallpositive integer,ϵ. And then continue the process of completing the Routh's table. Now, find the number of sign changes in the first column of the Routh's table by substituting $\epsilon\epsilon$tends to zero.

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristic equation,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Step1–VerifythenecessaryconditionfortheRouth-Hurwitzstability. All the coefficients of the characteristic polynomial,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

arepositive.So,thecontrolsystemsatisfiedthe necessarycondition.

Step2–FormtheRoutharrayforthegivencharacteristicpolynomial.

s^4	1	1	1
s^3	$\cancel{2}\ 1$	$\cancel{2}\ 1$	
s^2	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(0\times1)}{1} = 1$	
s^1			
s^0			

The row s^3elements have 2 as the common factor. So, all these elements are divided by 2.
Specialcase(i)–Onlythefirstelementofrows^{2}iszero.So,replaceitbyϵandcontinuethe process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon \times 1)-(1 \times 1)}{\epsilon} = \frac{\epsilon - 1}{\epsilon}$		
s^0	1		

Step3–Verifythe sufficientconditionfortheRouth-Hurwitzstability.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

As ϵ tends to zero, the Routh table becomes like this.

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

All the Elements of any row of the Routh's array are zero

In this case, follow these two steps−

8. Write the auxilary equation, A(s) of the row, which is just above the row of zeros.
9. Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example

Let us find the stability of the control system having characteristic equation,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Step1−Verifythe necessaryconditionfortheRouth-Hurwitzstability.

Allthecoefficientsofthegivencharacteristicpolynomialarepositive.So,thecontrolsystem satisfied the necessary condition.

Step2−FormtheRoutharrayforthegivencharacteristicpolynomial.

s^5	1	1	1
s^4	$\cancel{3}$ 1	$\cancel{3}$ 1	$\cancel{3}$ 1
s^3	$\frac{(1\times1)-(1\times1)}{1}=0$	$\frac{(1\times1)-(1\times1)}{1}=0$	
s^2			
s^1			
s^0			

The row s^4 elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) – All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4.

$$A(s) = s^4 + s^2 + 1$$

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = 4s^3 + 2s$$

Place these coefficients in row s^3.

s^5	1	1	1
s^4	1	1	1
s^3	$\cancel{4}$ 2	$\cancel{2}$ 1	
s^2	$\frac{(2\times1)-(1\times1)}{2} = 0.5$	$\frac{(2\times1)-(0\times1)}{2} = 1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5} = \frac{-1.5}{0.5}$ $= -3$		
s^0	1		

Step3–VerifythesufficientconditionfortheRouth-Hurwitzstability.

There are twosign changes in the first column of Routh table. Hence, the control system is unstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in onlefthalfofthe‘s’planeorontherighthalfofthe‘s’planeoronanimaginaryaxis.So,we can’tfindthenatureofthecontrolsystem.Toovercomethislimitation,thereisatechnique known as the root locus.

RootlocusTechnique

Inthe root locus diagram,we can observe the pathofthe closed looppoles. Hence,we can identifythenatureofthecontrolsystem.Inthistechnique,wewilluseanopenlooptransfer function to know the stability of the closed loop control system.

BasicsofRootLocus

TheRootlocusisthelocusoftherootsofthecharacteristicequationbyvaryingsystemgain K from zero to infinity.

Weknowthat,thecharacteristicequationoftheclosedloopcontrolsystemis

$$1 + G(s)H(s) = 0$$

We can represent $G(s)H(s)$ as

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

Where,

- K represents the multiplying factor
- N(s) represents the numerator term having (factored) n^{th} order polynomial of 's'.
- D(s) represents the denominator term having (factored) m^{th} order polynomial of 's'.

Substitute, $G(s)H(s)$ value in the characteristic equation.

$$1 + k\frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

Case 1 − K = 0

If $K = 0$, then $D(s) = 0$.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 − K = ∞

Re-write the above characteristic equation as

$$K\left(\frac{1}{K} + \frac{N(s)}{D(s)}\right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute, $K = \infty$ in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

If $K = \infty$, then $N(s) = 0$. It means the closed loop poles are equal to the open loop zeros when K is infinity.

Fromabovetwocases,wecanconcludethattherootlocusbranchesstartatopenlooppoles and end at open loop zeros.

AngleConditionandMagnitudeCondition

The points on the root locus branches satisfy the angle condition. So,the angle condition is used toknow whetherthe point exist onroot locus branch ornot. We can find the value of Kforthepointsontherootlocusbranchesbyusingmagnitudecondition.So,wecanusethe magnitude condition for the points, and this satisfies the angle condition.

Characteristicequationofclosedloopcontrolsystemis

$$1 + G(s)H(s) = 0$$
$$\Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of $G(s)H(s)$ is

$$\angle G(s)H(s) = \tan^{-1}\left(\frac{0}{-1}\right) = (2n+1)\pi$$

The**angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180^0.

MagnitudeofG(s)H(s)G(s)H(s)is–

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

Themagnitudecondition isthatthe point (whichsatisfiedtheanglecondition)at whichthe magnitude of the open loop transfers function is one.

The**rootlocus** isagraphicalrepresentationins-domainandit issymmetricalaboutthereal axis.Becausetheopenlooppolesandzerosexistinthes-domainhavingthevalueseitheras realorascomplexconjugatepairs.Inthischapter,letusdiscusshowtoconstruct(draw)the root locus.

RulesforConstructionofRootLocus

Followtheserulesforconstructingarootlocus.

Rule1−Locatetheopen looppolesandzerosin the‘s’plane.

Rule2−Findthenumberofrootlocus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches**N**is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically,wecanwritethenumberofroot locusbranches**N**as

N=PifP≥Z
N=ZifP<Z

Rule3−Identifyanddrawthe**realaxisrootlocusbranches**.

Iftheangleoftheopenlooptransferfunctionatapointisanoddmultipleof180^0,thenthat point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule4−Findthecentroidand theangleofasymptotes.

1. IfP=Z,thenalltherootlocusbranchesstartatfiniteopenlooppolesandendatfinite open loop zeros.

2. IfP>Z, then Z number of root locus branches start at finite open loop poles and end at finite open loopzeros andP−Znumber of root locus branches start at finite open loop poles and end at infinite open loop zeros.

3. IfP<Z,then P number of root locus branches start at finite open loop poles and end atfiniteopenloopzerosandZ−P numberofrootlocusbranchesstartatinfiniteopen loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when P≠Z. Asymptotes give the directionoftheseroot locusbranches.The intersectionpoint ofasymptotesontherealaxis is known as **centroid**.

Wecancalculatethe**centroidα**byusingthis formula,

$$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P-Z}$$

The formula for the angle of **asymptotes θ** is

$$\theta = \frac{(2q+1)180^0}{P-Z}$$

Where,

$$q = 0, 1, 2, \ldots, (P-Z) - 1$$

Rule5−Findtheintersectionpointsofrootlocusbrancheswithanimaginaryaxis.

We cancalculate the point at whichthe root locus branchintersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

4. If allelementsofanyrowof theRoutharrayarezero,then therootlocusbranch intersects the imaginary axis and vice-versa.

5. Identifytherowinsuchawaythatifwemakethefirstelementaszero, thenthe elements of the entire row are zero. Find the value of **K** for this combination.

6. SubstitutethisKvalueintheauxiliaryequation.Youwillgettheintersectionpoint of the root locus branch with an imaginary axis.

Rule6−FindBreak-awayandBreak-inpoints.

7. Ifthere exists a realaxis root locus branch betweentwoopenlooppoles,thenthere will be a **break-away point** in between these two open loop poles.

8. Ifthere exists a realaxis root locus branchbetweentwoopenloopzeros,thenthere will be a **break-in point** in between these two open loop zeros.

Note−Break-awayandbreak-inpointsexistonly onthe realaxisroot locusbranches.

Follow these steps to find break-away and break-in points.

9. WriteK intermsofsfrom thecharacteristicequation1+G(s)H(s)=0.

10. DifferentiateK with respect to s and makeit equal to zero. Substitute these valuesof ss in the above equation.

11. ThevaluesofssforwhichtheKvalueispositivearethe**breakpoints**.

Rule7−Findtheangleofdepartureandtheangleofarrival.

TheAngleofdepartureandtheangleofarrivalcanbecalculatedatcomplexconjugateopen loop poles and complex conjugate open loop zeros respectively.

Theformulaforthe**angleofdeparture**$\phi_d$is

$$\phi_d = 180^0 - \phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a = 180^0 + \phi$$

Where,

$$\phi = \sum \phi_P - \sum \phi_Z$$

Example

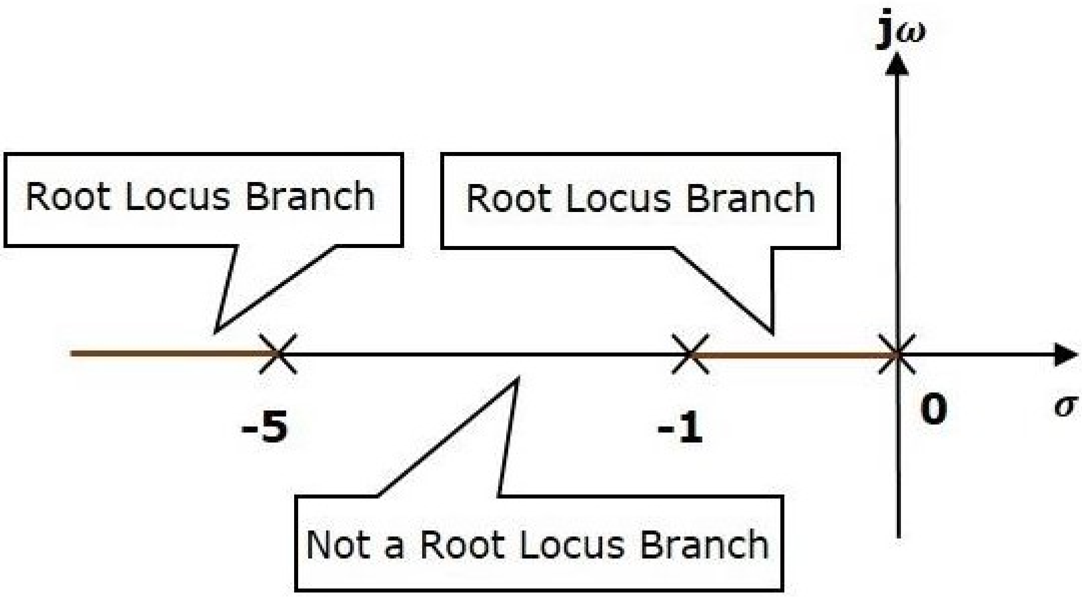
Letusnowdrawtherootlocusofthecontrolsystemhavingopenlooptransfer

function, $G(s)H(s) = \frac{K}{s(s+1)(s+5)}$

Step1−Thegivenopenlooptransferfunctionhasthreepolesats=0,

s=-1,s=-5.Itdoesn’thaveanyzero.Therefore,thenumberofrootlocusbranchesisequal to the number of poles of the open loop transfer function.

N=P=3



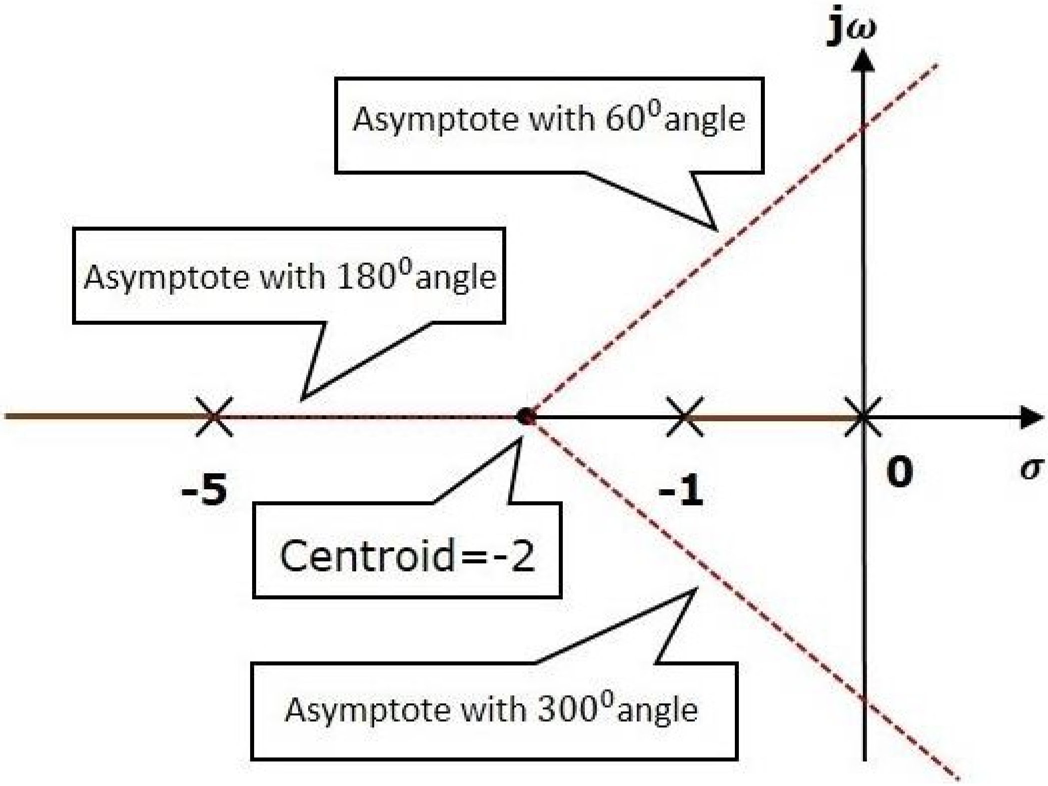
Thethreepoles are located are showninthe abovefigure.The line segmentbetween s=−1, ands=0is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of s=−5.

Step 2− We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid

Theangleofasymptotesare $\theta = 60^0, 180^0$ and 300^0.

Thecentroidandthreeasymptotesareshowninthefollowing figure.



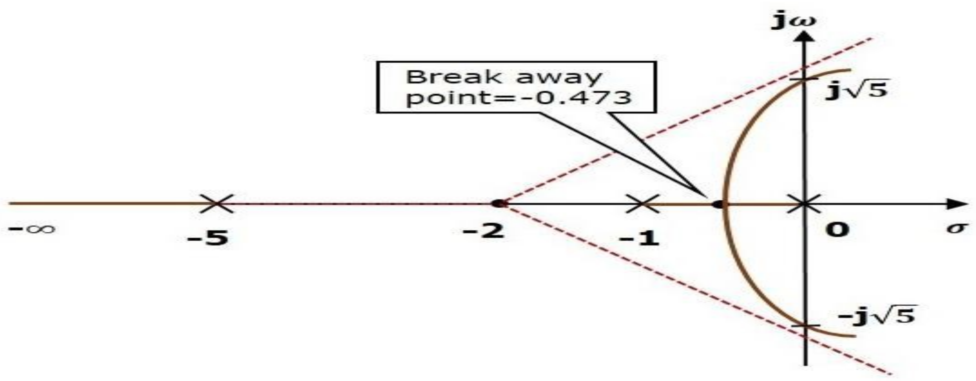
Step3−Sincetwoasymptoteshavetheanglesof 600600and30003000,tworootlocus branchesintersecttheimaginaryaxis. Byusing theRoutharraymethodandspecialcase(ii),

theroot locusbranchesintersectstheimaginaryaxisat $j\sqrt{5}$ and $-j\sqrt{5}$.

Therewillbeonebreak-awaypointontherealaxisrootlocusbranchbetweenthepoless

=−1ands=0. By following the procedure given for the calculation of break-away point, we will get it as s =−0.473.

Therootlocusdiagramforthegivencontrolsystemisshowninthe following figure.



In thisway,youcandrawtherootlocusdiagramofanycontrolsystemand observethe movement of poles of the closed loop transfer function.

Fromtherootlocusdiagrams,wecanknowtherangeofKvaluesfordifferenttypesof damping.

EffectsofAddingOpenLoopPolesandZerosonRootLocus

The root locus can be shifted in **'s' plane**by adding the open loop poles and the open loop zeros.

1. If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of 's' plane. Because of this, the dampingratioδdecreases. Which implies, damped frequency ωdincreases and the time domainspecifications like delay time td,rise timetrandpeak timetpdecrease. But, it effects the system stability.

2. If we include a zero in the open loop transfer function, then some of root locus brancheswillmovetowardslefthalfof's'plane.So,itwillincreasethecontrolsystem stability. In this case, the damping ratio δ increases. Which implies, damped frequencyωddecreases and the time domain specifications like delay timetd, rise time tr and peak time tp increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

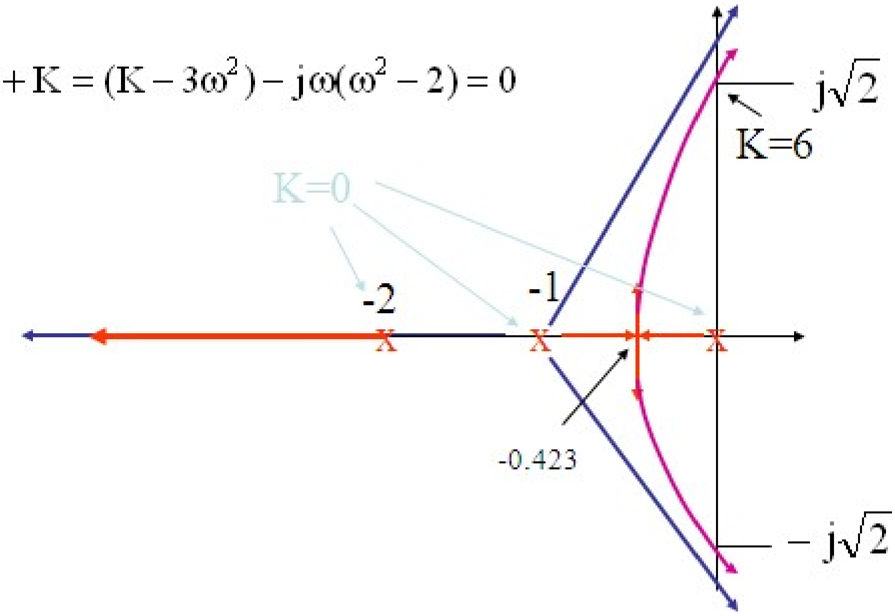
Example

$$\boxed{s^3 + 3s^2 + 2s + K = 0}$$

$$s = j\omega$$

$$-j\omega^3 - 3\omega^2 + 2j\omega + K = (K - 3\omega^2) - j\omega(\omega^2 - 2) = 0$$

$$\boxed{\begin{aligned} &\omega^2 = 2 \\ &K = 3\omega^2 = 6 \end{aligned}}$$



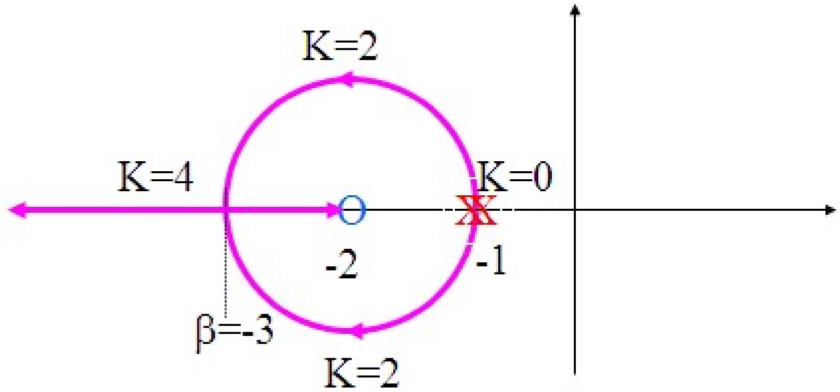
Example

$$\boxed{GH(s) = \frac{K(s+2)}{(s+1)^2}}$$

Two poles at –1
One zero at –2
One asymptote at 180°
Break-in point at -3

$$\frac{2}{\beta+1} = \frac{1}{\beta+2}$$

$$2\beta + 4 = \beta + 1$$



Example Why a circle ?

Characteristic equation $s^2 + s(2+K) + 2K + 1 = 0$

For K<4

$$s_{1,2} = \frac{-(2+K) \pm j\sqrt{K(4-K)}}{2}$$

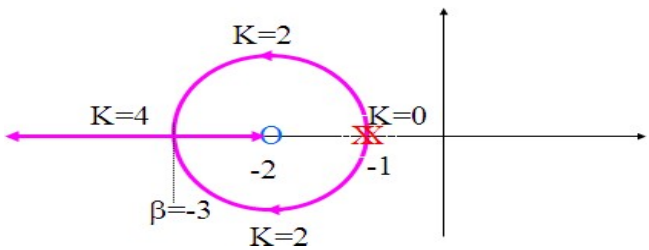
For K>4

$$s_{1,2} = \frac{-(2+K) \pm \sqrt{K(K-4)}}{2}$$

Change of origin $s_{1,2} + 2 = \frac{-(-2+K) \pm j\sqrt{K(4-K)}}{2}$

$$4m = (K-2)^2 + K(4-K) = K^2 - 4K + 4 + 4K - K^2$$

$$m = 1$$



Effectsofaddingapoleorazerotothe root locusofasecond-ordersystem

We discussed how we could change the value of gain K to change the position of the closed-looppoles.Thiscorrespondstoplacingaproportionalgain, K,incascadewiththe system $G(s)$ and finding the closed-loop poles for different values of gain, K. However, proportional control is a simple form of control; it does not provide us with zero steady example, in some control design problems, to produce the performance required in the designspecificationsweneedtomovethepolestosomepositionsonthe s-plane,which maynotlieonarootlocusdefinedbythesimpleproportionalgain K.Tobeabletomove the poles to any positiononthe s-plane, we need touse a more complicated controller. Forexample,wemayneedtoadd azero ora pole tothecontrollerand seehowthiswill affect the root locus and hence the position of the closed-loop poles. Examples of controllers with poles or zeros are:

PI control: $$K(s) = \mathrm{K_p} + \frac{K_\mathrm{i}}{s} = \frac{K_\mathrm{p}s + K_\mathrm{i}}{s}$$

Lag controller: $$K(s) = \frac{s\tau+1}{\alpha s\tau+1} \qquad (\tau,\ \alpha \text{ are controller parameters})$$

Thus, weneed toknow howtherootlocus will changeifweadd apoleor azero. To investigate this, we will use a simple example.

Effectsofaddinga zeroontherootlocusforasecond-order system

Considerthesecond-ordersystemgiven by

$$G(s) = \frac{1}{(s+p_1)(s+p_2)} \qquad p_1 > 0, \qquad p_2 > 0$$

The poles are given by $s = -p1$ and $s = -p2$ and the simple root locus plot for this system is shownin Figure 13.13(a). Whenwe add a zeroat $s= -z1$ to the controller, the open-loop transfer function will change to:

$$G_1(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)}, \qquad z_1 > 0$$

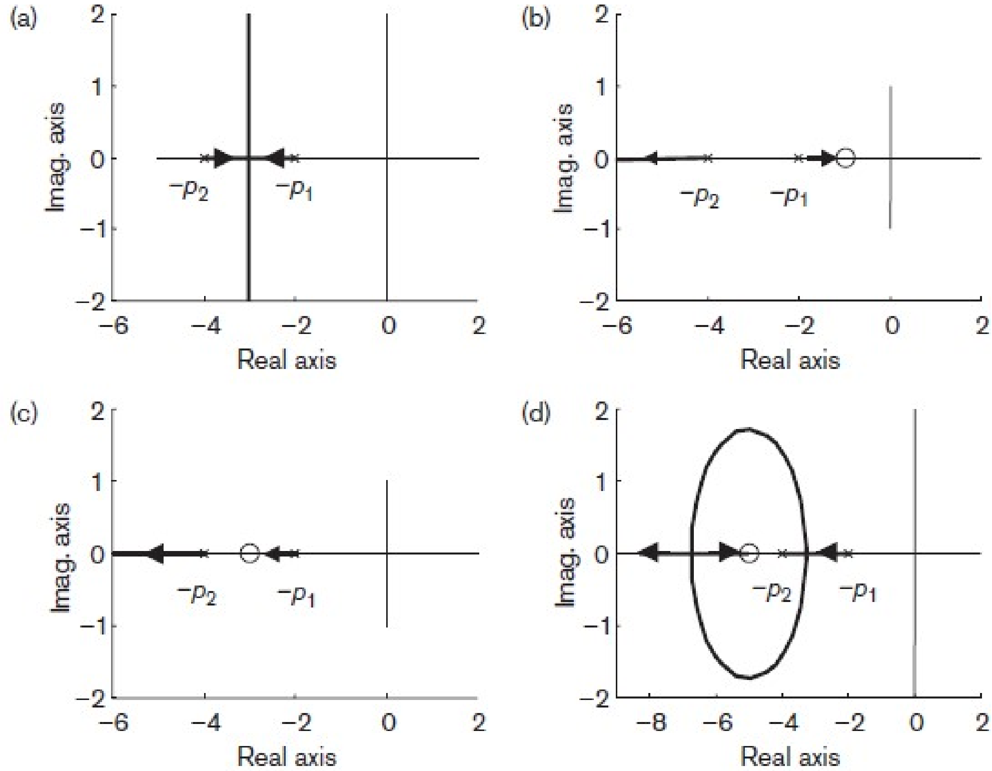


Figure · Effect of adding a zero to a second-order system root locus.

UNIT-IV

FREQUENCYRESPONSEANALYSIS

WhatisFrequencyResponse?

Theresponseofasystemcanbepartitionedintoboththetransientresponseandthesteady state response. We can find the transient response by using Fourier integrals. The steady stateresponseofasystemforaninputsinusoidalsignalisknownasthe**frequencyresponse**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles. Let the input signal be

$$r(t) = A\sin(\omega_0 t)$$

The open loop transfer function will be −

$$G(s) = G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

Substitute, $\omega = \omega_0$ in the above equation.

$$G(j\omega_0) = |G(j\omega_0)|\angle G(j\omega_0)$$

The output signal is

$$c(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega = \omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega = \omega_0$.

Where,

1. **A**istheamplitude oftheinputsinusoidalsignal.

2. **ω_0**isangularfrequencyoftheinputsinusoidalsignal.

We can write, angular frequency $\omega_0$as shown below.

$$\omega_0=2\pi f_0$$

Here, f_0 is the frequency of the input sinusoidal signal. Similarly, you can follow the same procedure for closed loop control system.

FrequencyDomainSpecifications

Thefrequencydomainspecificationsare

3. **Resonantpeak**
4. **Resonantfrequency**
5. **Bandwidth**.

Considerthetransferfunctionofthesecondorderclosedcontrolsystemas

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Substitute, $s = j\omega$ in the above equation.

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow T(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega\omega_n + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\frac{\omega}{\omega_n} = u$ Substitute this value in the above equation.

$$T(j\omega) = \frac{1}{(1-u^2) + j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1-u^2}\right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r. At $\omega = \omega_r$, the first derivate of the magnitude of $T(j\omega)$ is zero.

Differentiate M with respect to u.

$$\frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[2(1-u^2)(-2u) + 2(2\delta u)(2\delta)\right]$$

$$\Rightarrow \frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[4u(u^2 - 1 + 2\delta^2)\right]$$

Substitute, $u = u_r$ and $\frac{\mathrm{d}M}{\mathrm{d}u} == 0$ in the above equation.

$$0 = -\frac{1}{2}\left[(1-u_r^2)^2 + (2\delta u_r)^2\right]^{-\frac{3}{2}}\left[4u_r(u_r^2 - 1 + 2\delta^2)\right]$$

$$\Rightarrow 4u_r(u_r^2 - 1 + 2\delta^2) = 0$$

$$\Rightarrow u_r^2 - 1 + 2\delta^2 = 0$$

$$\Rightarrow u_r^2 = 1 - 2\delta^2$$

$$\Rightarrow u_r = \sqrt{1 - 2\delta^2}$$

Substitute, $u_r = \frac{\omega_r}{\omega_n}$ in the above equation.

$$\frac{\omega_r}{\omega_n} = \sqrt{1 - 2\delta^2}$$

$$\Rightarrow \omega_r = \omega_n\sqrt{1 - 2\delta^2}$$

ResonantPeak

Itisthepeak(maximum)valueofthemagnitudeofT(jω).Itis denotedbyM$_r$. At u=u$_r$,

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1-2\delta^2}$ and $1 - u_r^2 = 2\delta^2$ in the above equation.

$$M_r = \frac{1}{\sqrt{(2\delta^2)^2 + (2\delta\sqrt{1-2\delta^2})^2}}$$

the Magnitude of $T(j\omega)$ is -

$$\Rightarrow M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

Resonantpeakinfrequencyresponsecorrespondstothepeakovershootinthetimedomain transient response for certain values of damping ratio δδ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth

Itistherangeoffrequenciesoverwhich,themagnitudeofT(jω)dropsto70.7%fromitszero frequency value.

Atω=0,thevalueofuwillbezero. Substitute,
u=0 in M.

$$M = \frac{1}{\sqrt{(1-0^2)^2 + (2\delta(0))^2}} = 1$$

Therefore,themagnitudeofT(jω)isoneatω=0

At3-dB frequency,themagnitudeofT(jω) willbe70.7%ofmagnitudeofT(jω))atω=0

i.e., at $\omega = \omega_B, M = 0.707(1) = \frac{1}{\sqrt{2}}$

$$\Rightarrow M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\delta u_b)^2}}$$

$$\Rightarrow 2 = (1-u_b^2)^2 + (2\delta)^2 u_b^2$$

Let, $u_b^2 = x$

$$\Rightarrow 2 = (1-x)^2 + (2\delta)^2 x$$

$$\Rightarrow x^2 + (4\delta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x.

$$x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}$$

$$\Rightarrow x = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

Substitute, $x = u_b^2 = \frac{\omega_b^2}{\omega_n^2}$

$$\frac{\omega_b^2}{\omega_n^2} = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

$$\Rightarrow \omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}}$$

Bandwidthωbin the frequency response is inversely proportional to the rise time tr in the time domain transient response.

Bodeplots

TheBodeplotortheBodediagramconsistsoftwoplots−

1. Magnitudeplot
2. Phaseplot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, yaxis representsthemagnitude(linearscale)ofopenlooptransferfunctioninthemagnitudeplot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The**magnitude**ofthe openlooptransferfunctionindBis-

$$M = 20\ \log|G(j\omega)H(j\omega)|$$

The**phaseangle**oftheopenlooptransferfunctionindegreesis-

$$\phi = \angle G(j\omega)H(j\omega)$$

BasicofBodePlots

The following table shows the slope, magnitude and the phase angle values of the terms presentinthe open looptransferfunction.Thisdataisusefulwhile drawingtheBode plots.

Type of term	G(jω)H(jω)	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20\log K$	0
Zero at origin	$j\omega$	20	$20\log\omega$	90
'n' zeros at origin	$(j\omega)^n$	$20\,n$	$20\,n\log\omega$	$90\,n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20\log\omega$	$-90\ or\ 270$
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20\,n$	$-20\,n\log\omega$	$-90\,n\ or\ 270\,n$
Simple zero	$1+j\omega r$	20	$0\ for\ \omega < \frac{1}{r}$ $20\ \log\omega r\ for\ \omega > \frac{1}{r}$	$0\ for\ \omega < \frac{1}{r}$ $90\ for\ \omega > \frac{1}{r}$

Simple pole	$\frac{1}{1+j\omega r}$	-20	$0\ for\ \omega < \frac{1}{r}$ $-20\ \log \omega r\ for\ \omega > \frac{1}{r}$	$0\ for\ \omega < \frac{1}{r}$ $-90\ or\ 270\ for\ \omega > \frac{1}{r}$
Second order derivative term	$\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)$	40	$40\ \log\ \omega_n\ for\ \omega < \omega_n$ $20\ \log\ (2\delta\omega_n^2)\ for\ \omega = \omega_n$ $40\ \log\ \omega\ for\ \omega > \omega_n$	$0\ for\ \omega < \omega_n$ $90\ for\ \omega = \omega_n$ $180\ for\ \omega > \omega_n$
Second order integral term	$\frac{1}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)}$	-40	$-40\ \log\ \omega_n\ for\ \omega < \omega_n$ $-20\ \log\ (2\delta\omega_n^2)\ for\ \omega = \omega_n$ $-40\ \log\ \omega\ for\ \omega > \omega_n$	$-0\ for\ \omega < \omega_n$ $-90\ for\ \omega = \omega_n$ $-180\ for\ \omega > \omega_n$

Consider the open loop transfer function $G(s)H(s) = K$.

Magnitude $M = 20\ \log K$ dB

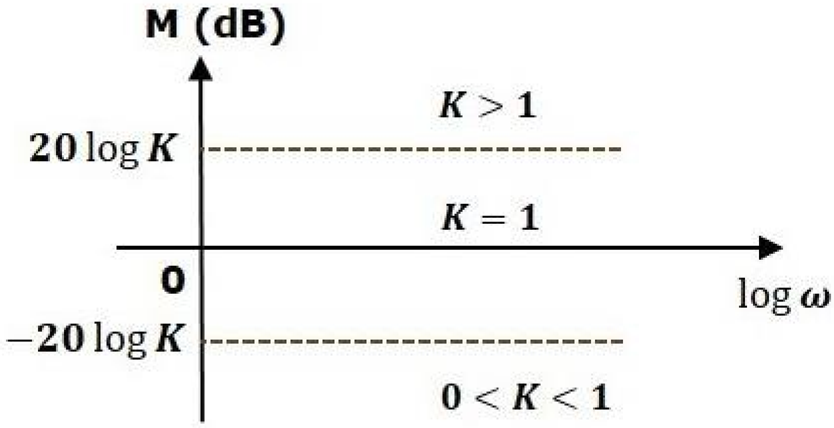
Phase angle $\phi = 0$ degrees

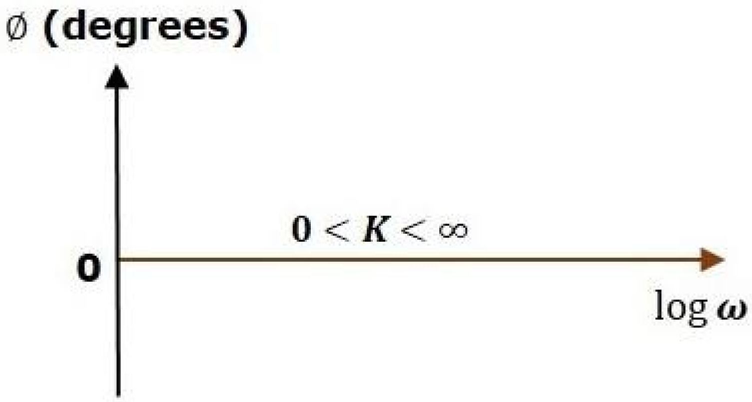
If $K = 1$, then magnitude is 0 dB.

If $K > 1$, then magnitude will be positive.

If $K < 1$, then magnitude will be negative.

The following figure shows the corresponding Bode plot.





The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift 20logK dB above the 0 dB line. For the negative values of K, the horizontal line will shift 20logK dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.

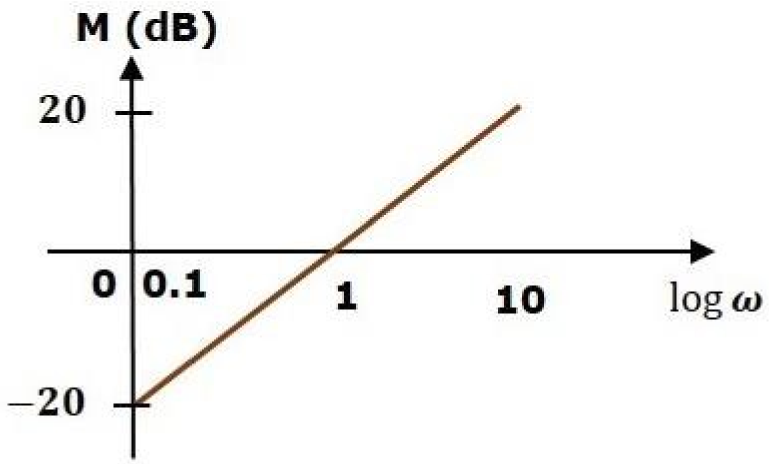
ConsidertheopenlooptransferfunctionG(s)H(s)=s
Magnitude M=20logω dB

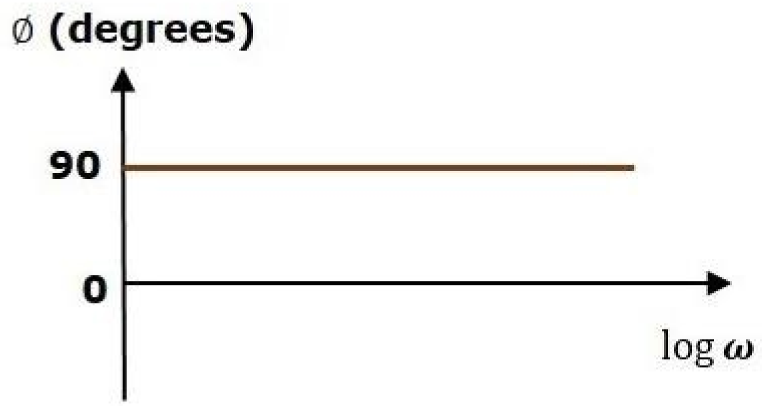
Phaseangleϕ=90^0

Atω=0.1rad/sec,themagnitudeis-20dB. At ω=1rad/sec, the magnitude is 0 dB.

Atω=10rad/sec,themagnitudeis20 dB.

ThefollowingfigureshowsthecorrespondingBodeplot.





Themagnitudeplotisaline,whichishavingaslopeof20dB/dec.Thislinestarted atω=0.1rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at ω=1 rad/sec. In this case, the phase plot is 90^0 line.

ConsidertheopenlooptransferfunctionG(s)H(s)=1+sτ.

Magnitude $M = 20\ log\sqrt{1+\omega^2\tau^2}$ dB

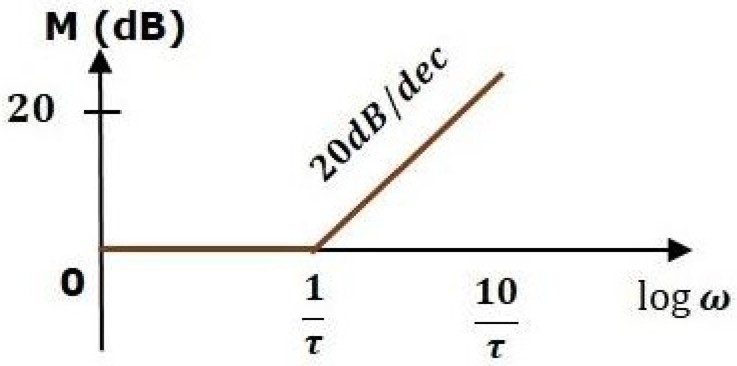
$$\phi = \tan^{-1}\omega\tau \text{ degrees}$$

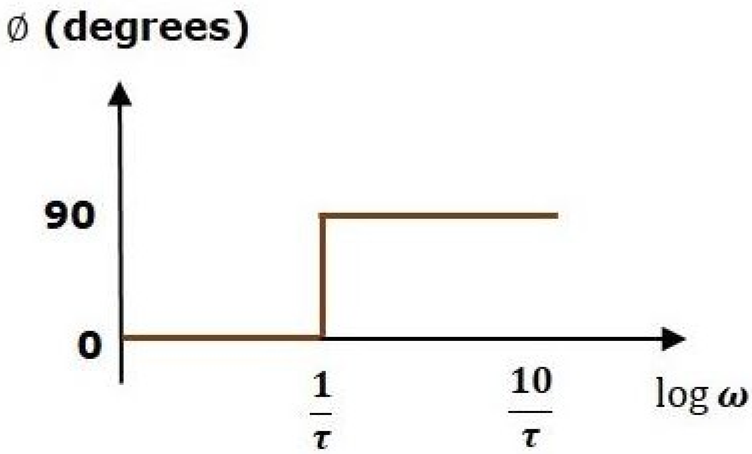
Phaseangle

$$\omega < \frac{1}{\tau}$$

For ,themagnitudeis0dB andphaseangle is0degrees.

For $\omega > \frac{1}{\tau}$,themagnitudeis 20logωτdBandphaseangleis90^0. The following figure shows the corresponding Bode plot





Themagnitudeplotishavingmagnitudeof0dBuptoω=1τω=1τrad/sec.From ω=1τrad/sec, itishavingaslopeof20dB/dec.Inthiscase,thephaseplotishavingphaseangleof0degrees up to ω=1τ rad/sec and from here, it is having phase angle of 90^0. This Bode plot is called the **asymptotic Bode plot**.

As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Similarly, you can draw the Bode plots for other terms of the open loop transfer function which are given in the table.

RulesforConstructionofBodePlots

FollowtheseruleswhileconstructingaBode plot.

1. Representtheopenlooptransferfunctioninthestandardtimeconstantform.

2. Substitute,s=jωs=jωintheaboveequation.
3. Findthecornerfrequenciesandarrangetheminascendingorder.
4. Consider the starting frequency of the Bode plot as 1/10thof the minimum corner frequency or0.1rad/secwhicheveris smallervalue anddraw the Bode plot upto10 times maximum corner frequency.
5. Drawthemagnitudeplotsforeachtermandcombinetheseplots properly.
6. Drawthephaseplotsforeachtermandcombinetheseplotsproperly.

Note– The cornerfrequency is the frequency at whichthere is a change inthe slope of the magnitude plot.

Example

Considertheopen looptransferfunctionofaclosedloopcontrolsyste

$$G(s)H(s) = \frac{10s}{(s+2)(s+5)}$$

Let us convert this open loop transfer function into standard time constant form.

$$G(s)H(s) = \frac{10s}{2\left(\frac{s}{2}+1\right)5\left(\frac{s}{5}+1\right)}$$

$$\Rightarrow G(s)H(s) = \frac{s}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)}$$

So, we can draw the Bode plot in semi log sheet using the rules mentioned earlier.

StabilityAnalysisusingBodePlots

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

7. Gaincrossoverfrequencyandphasecrossoverfrequency
8. Gainmarginandphasemargin

PhaseCrossoverFrequency

The frequency at whichthe phase plot is having the phase of -180^0 is knownas **phasecross over frequency**. It is denoted by ωpc. The unit of phase cross over frequency is **rad/sec**.

GainCrossoverFrequency

Thefrequencyatwhichthemagnitudeplotishavingthemagnitudeofzero dBisknown as**gaincrossoverfrequency**.Itisdenotedby ωgc.Theunitofgaincrossoverfrequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

9. Ifthephasecrossoverfrequency ωpc isgreaterthanthegaincrossover frequency ωgc, then the control system is **stable**.
10. If the phase cross over frequency $\omega pc$is equal to the gain cross over frequency ωgc, then the control system is **marginally stable**.
11. Ifthephasecrossoverfrequencyωpcislessthanthegaincrossesoverfrequencyωgc, then the control system is **unstable**.

GainMargin

GainmarginGMGMisequaltonegativeofthemagnitudeindBatphasecrossover frequency.

$$GM=20\log(1Mpc)=20\log Mpc$$

Where,MpcMpc is the magnitude at phase cross over frequency. The unit of gain margin (GM) is **dB**.

PhaseMargin

TheformulaforphasemarginPMPMis

$$PM=180^{0}+\boldsymbol{\phi} gc$$

Where,**ϕ**gcisthephaseangleatgaincrossoverfrequency.Theunitofphasemargin is **degrees**.

NOTE:

The stability of the control system based on the relation between gain margin and phasemargin is listed below.

12. If both the gain marginGMand the phase marginPMare positive, then the control system is **stable**.

13. IfboththegainmarginGMandthephasemarginPMareequaltozero,thenthe control system is **marginally stable**.

    IfthegainmarginGMand/orthephasemarginPMare/isnegative,thenthecontrol system is **unstable**.

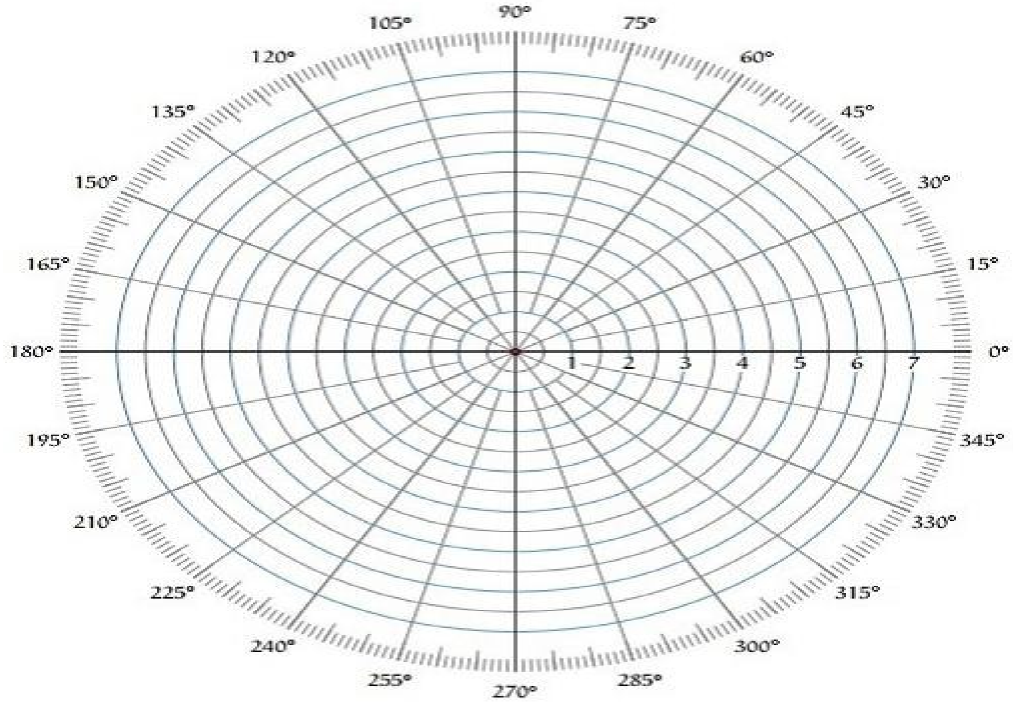
    Polarplots

Polar plotis a plot which can bedrawn between magnitude and phase. Here, themagnitudes are represented by normal values only.

The polar form of $G(j\omega)H(j\omega)$ is

$$G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)$$

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of $G(j\omega)H(j\omega)$ by varying ω from zero to ∞. The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circles and radial lines. The **concentric circles** and the**radial lines**represent the magnitudes and phase angles respectively. These angles are representedbypositivevaluesinanti-clockwisedirection.Similarly,wecanrepresentangles with negative values in clockwise direction. For example, the angle 270^0in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

RulesforDrawingPolarPlots

Followtheserulesforplottingthepolar plots.

14. Substitute,s=jωinthe openlooptransferfunction.

15. Writetheexpressionsformagnitude andthe phaseofG(jω)H(jω)

16. Findthestartingmagnitudeandthe phaseof $G(j\omega)H(j\omega)$bysubstituting $\omega=0$.So,the polar plot starts with this magnitude and the phase angle.

17. Find the ending magnitude andthe phase of G(jω)H(jω) by substituting ω=∞ So,the polar plot ends with this magnitude and the phase angle.

18. Check whether the polar plot intersects the real axis, by making the imaginary term of G(jω)H(jω) equal to zero and find the value(s) of ω.

19. Checkwhetherthepolarplotintersectstheimaginaryaxis,bymakingrealterm of G(jω)H(jω) equal to zero and find the value(s) of ω.

20. For drawing polar plot more clearly, find the magnitude and phase of G(jω)H(jω) by considering the other value(s) of ω.

Example

Considertheopen looptransferfunctionofaclosedloopcontrolsystem.

$$G(s)H(s) = \frac{5}{s(s+1)(s+2)}$$

Let us draw the polar plot for this control system using the above rules.

Step 1 − Substitute, $s = j\omega$ in the open loop transfer function.

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

The magnitude of the open loop transfer function is

$$M = \frac{5}{\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})}$$

The phase angle of the open loop transfer function is

$$\phi = -90^0 - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}$$

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	∞	-90 or 270
∞	0	-270 or 90

So, the polar plot starts at $(\infty,-90^0)$ and ends at $(0,-270^0)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

Step3–Basedonthestartingandtheendingpolarco-ordinates,thispolarplotwillintersect the negative real axis. The phase angle corresponding to the negative real axis is -180^0or 180^0. So, by equating the phase angle of the open loop transfer function to either -180^0 or 180^0, we will get the ω value as √2.

Bysubstituting $\omega=\sqrt{2}$inthemagnitudeoftheopenlooptransferfunction,wewill getM=0.83. Therefore, the polar plot intersects the negative real axis when $\omega=\sqrt{2}$and the polar coordinate is $(0.83,-180^0)$.

So,wecan drawthepolarplotwiththeaboveinformationonthepolargraphsheet.

NyquistPlots

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop controlsystemsbyvaryingωfrom−∞to∞.Thatmeans,Nyquistplotsareusedtodrawthe complete frequency response of the open loop transfer function.

NyquistStabilityCriterion

TheNyquist stability criterionworks onthe **principleof argument**. It statesthat ifthereare PpolesandZzerosareenclosedbythe‘s’planeclosedpath,thenthe corresponding G(s)H(s)G(s)H(s)plane must encircle the originP−ZP−Ztimes. So, we can write the number of encirclements N as,

$$N=P-ZN=P-Z$$

21. If the enclosed ‘s’ plane closed path contains only poles, then the direction of the encirclement in the G(s)H(s)G(s)H(s)plane will be opposite to the direction of the enclosed closed path in the ‘s’ plane.

22. If the enclosed ‘s’ plane closed path contains only zeros, then the direction of the encirclementintheG(s)H(s)G(s)H(s)planewillbeinthesamedirectionasthatofthe enclosed closed path in the ‘s’ plane.

Letusnowapplytheprincipleofargumenttotheentirerighthalfofthe‘s’planebyselecting it as a closed path. This selected path is called the **Nyquist** contour.

We know that the closed loop control system is stable if all the poles of the closed loop transferfunctionare inthe left halfof the‘s’plane. So,the poles ofthe closed looptransfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

23. ThePolesofthecharacteristicequationaresameasthatofthepolesoftheopenloop transfer function.

24. The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

Weknowthattheopenloopcontrolsystemisstableifthere isnoopenlooppole inthe the right half of the ‘s’ plane.

i.e.,$P=0 \Rightarrow N=-Z$$P=0 \Rightarrow N=-Z$

We know that the closed loop controlsystem is stable if there is no closed loop pole in the right half of the ‘s’ plane.

i.e.,Z=0⇒N=PZ=0⇒N=P

Nyquiststabilitycriterionstatesthenumberofencirclementsaboutthecriticalpoint(1+j0) must be equalto the poles of characteristic equation,which is nothing but the poles ofthe openlooptransferfunctionintherighthalfofthe‘s’plane.Theshiftinoriginto(1+j0)gives the characteristic equation plane.

RulesforDrawingNyquistPlots

FollowtheserulesforplottingtheNyquistplots.

1. Locatethepolesandzerosofopenlooptransferfunction G(s)H(s)in‘s’ plane.
2. Drawthepolarplotbyvarying ωfromzerotoinfinity.Ifpole orzeropresentats=0, then varying ω from 0+ to infinity for drawing polar plot.
3. Draw the mirror image of above polar plot for values of ω ranging from −∞ to zero (0^- if any pole or zero present at s=0).
4. Thenumberofinfiniteradiushalfcircleswillbeequaltothenumberofpolesorzeros at origin. The infinite radius halfcircle willstart at the point where the mirror image ofthe polar plot ends. Andthis infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system usingtheNyquiststabilitycriterion.Ifthecritical point (-1+j0)liesoutsidetheencirclement, then the closed loop control system is absolutely stable.

StabilityAnalysisusingNyquistPlots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

5. Gaincrossoverfrequencyandphasecrossoverfrequency
6. Gainmarginandphasemargin

PhaseCrossover Frequency

ThefrequencyatwhichtheNyquistplotintersectsthenegativerealaxis(phaseangleis180^0) is known as the **phase cross over frequency**. It is denoted by ω_{pc}.

GainCrossover Frequency

ThefrequencyatwhichtheNyquistplotishavingthemagnitudeofoneisknownas the**gain cross over frequency**. It is denoted by ωgc.

Thestabilityofthecontrolsystembasedontherelationbetweenphasecrossoverfrequency and gain cross over frequency is listed below.

7. Ifthephasecrossoverfrequency ωpcisgreaterthanthegaincrossoverfrequency ωgc, then the control system is **stable**.

8. Ifthe phase cross over frequency ωpc is equal to the gain cross over frequency ωgc, then the control system is **marginally stable**.

9. Ifphasecrossoverfrequency ωpc islessthangaincrossoverfrequencyωgc,thenthe control system is **unstable**.

GainMargin

The gain marginGMis equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

Where,Mpc isthemagnitudeinnormalscaleatthephasecrossoverfrequency.

PhaseMargin

Thephasemargin PMisequaltothesum of180^0 andthephaseangleatthegaincrossover frequency.

$$PM=180^0+\boldsymbol{\phi}_{gc}$$

Where,$\boldsymbol{\phi}_{gc}$isthephaseangleatthegaincrossoverfrequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

10. If the gain marginGMis greater than one and the phase margin PM is positive, then the control system is **stable**.

11. If the gain marginGMs equal to one and the phase margin PMis zero degrees, then the control system is **marginally stable**.

12. IfthegainmarginGMislessthanoneand/orthephasemarginPMisnegative,then the control system is **unstable**.

UNIT-V

STATESPACEANALYSISOF CONTINUOUS SYSTEMS

The**state spacemodel**ofLinearTime-Invariant(LTI)systemcanberepresented as,

$$\dot{X}=AX+BU$$

$$Y=CX+DU$$

Thefirstandthesecondequationsareknownasstate equationandoutputequationrespectively.

Where,

13. XandẊarethestatevectorand thedifferentialstatevectorrespectively.
14. Uand Yareinputvectorandoutputvector respectively.
15. Ais thesystemmatrix.
16. BandCaretheinputandtheoutput matrices.
17. Disthefeed-forwardmatrix.

BasicConceptsofStateSpaceModel

Thefollowingbasicterminologyinvolvedinthischapter.

State

Itisagroupofvariables,whichsummarizesthehistoryofthesysteminordertopredictthe future values (outputs).

StateVariable

The number of the state variables required is equalto the number ofthe storage elements present in the system.

Examples−currentflowingthroughinductor,voltageacross capacitor

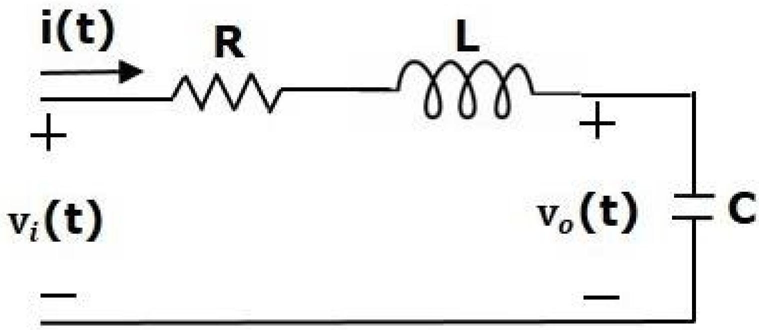
StateVector

Itisavector,whichcontainsthestatevariablesaselements.

Intheearlierchapters,wehavediscussedtwomathematicalmodelsofthecontrolsystems. Those are the differential equationmodeland the transferfunctionmodel. The state space modelcanbeobtainedfromanyoneofthesetwomathematicalmodels.Letusnowdiscuss these two methods one by one.

StateSpaceModelfromDifferentialEquation

Consider the following series of the RLC circuit. It is having an input voltage, vi(t)and the current flowing through the circuit is i(t).



There are two storage elements (inductor and capacitor) in this circuit. So, the number of thestatevariables isequaltotwoand thesestatevariablesare thecurrent flowingthrough the inductor, i(t) and the voltage across capacitor, $v_c(t)$.

Fromthecircuit,theoutputvoltage,$v_0(t)$isequaltothevoltageacrosscapacitor, $v_c(t)$.

$$v_0(t) = v_c(t)$$

Apply KVL around the loop.

$$v_i(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + v_c(t)$$

$$\Rightarrow \frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

The voltage across the capacitor is -

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{i(t)}{C}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix}$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \begin{bmatrix} v_i(t) \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},\ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}\ and\ D = \begin{bmatrix} 0 \end{bmatrix}$$

StateSpaceModelfromTransferFunction

Considerthetwotypesoftransferfunctionsbasedonthetypeoftermspresentinthe numerator.

18. TransferfunctionhavingconstantterminNumerator.
19. Transferfunctionhavingpolynomialfunctionof‘s’inNumerator.

Transfer function having constant term in Numerator

Considerthefollowingtransferfunctionofasystem

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0 U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}y(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1}y(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} = \dot{x}_n$$

Andu(t)=u

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1x_2 + a_0x_1 = b_0u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + b_0u$$

The output equation is -

$$y(t) = y = x_1$$

The state space model is -

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = [1 \quad 0 \quad \ldots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here,D=[0].

Example:

Findthestatespacemodelforthesystemhavingtransferfunction.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Rearrange, the above equation as,

$$(s^2 + s + 1)Y(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_2 = -x_1 - x_2 + u$$

The output equation is

$$y(t) = y = x_1$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transferfunctionhavingpolynomialfunctionof‘s’inNumerator Consider
the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s + b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \left(\frac{1}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s + a_0}\right)(b_n s^n + b_{n-1}s^{n-1}+\ldots+b_1 s + b_0)$$

The above equation is in the form of product of transfer functions of two blocks, which are cascaded.

$$\frac{Y(s)}{U(s)} = \left(\frac{V(s)}{U(s)}\right)\left(\frac{Y(s)}{V(s)}\right)$$

Here,

$$\frac{V(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1}+\ldots+a_1 s + a_0}$$

Rearrange,theaboveequation as

$$(s^n + a_{n-1}s^{n-1}+\ldots+a_0)V(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}}+\ldots+a_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + a_0 v(t) = u(t)$$

Let

$$v(t) = x_1$$

$$\frac{\mathrm{d}v((t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} = \dot{x}_n$$

andu(t)=u

Then,thestateequationis

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u$$

Consider,

$$\frac{Y(s)}{V(s)} = b_ns^n + b_{n-1}s^{n-1} + \ldots + b_1s + b_0$$

Rearrange, the above equation as

$$Y(s) = (b_ns^n + b_{n-1}s^{n-1} + \ldots + b_1s + b_0)V(s)$$

Apply inverse Laplace transform on both the sides.

$$y(t) = b_n\frac{\mathrm{d}^nv(t)}{\mathrm{d}t^n} + b_{n-1}\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}} + \ldots + b_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + b_0v(t)$$

By substituting the state variables and $y(t) = y$ in the above equation, will get the output equation as,

$$y = b_n\dot{x}_n + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

Substitute, $\dot{x}_n$ value in the above equation.

$$y = b_n(-a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u) + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

$$y = (b_0 - b_na_0)x_1 + (b_1 - b_na_1)x_2 + \ldots + (b_{n-1} - b_na_{n-1})x_n + b_nu$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} b_0 - b_na_0 & b_1 - b_na_1 & \ldots & b_{n-2} - b_na_{n-2} & b_{n-1} - b_na_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If $b_n = 0$, then,

$$Y = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

TransferFunctionfromStateSpaceModel

WeknowthestatespacemodelofaLinearTime-Invariant(LTI)systemis -

$$\dot{X}=AX+BU$$

$$Y=CX+DU$$

ApplyLaplaceTransformonbothsidesofthestateequation.

$$sX(s)=AX(s)+BU(s)$$

$$\Rightarrow (sI-A)X(s)=BU(s)$$

$$\Rightarrow X(s)=(sI-A)^{-1}BU(s)$$

ApplyLaplaceTransformonbothsidesoftheoutputequation.

$$Y(s)=CX(s)+DU(s)$$

Substitute,X(s)valueintheaboveequation.

$$\Rightarrow Y(s)=C(sI-A)^{-1}BU(s)+DU(s)$$

$$\Rightarrow Y(s)=[C\,(sI-A)^{-1}B+D]U(s)$$

$$\Rightarrow Y(s)U(s)=C(sI-A)^{-1}B+D$$

Theaboveequationrepresentsthetransferfunctionofthesystem.So,wecancalculatethe transferfunctionofthesystembyusingthisformulaforthesystemrepresentedinthestate space model.

Note–WhenD=[0],the transferfunctionwill be

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Example:

Letuscalculatethetransferfunction ofthesystemrepresentedinthestatespacemodel as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore,thetransferfunctionofthesystemforthegivenstatespacemodelis

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

StateTransitionMatrixanditsProperties

Ifthe system is having initial conditions,then it willproduce an output. Since,this output is presentevenintheabsenceofinput,itiscalled **zeroinputresponse**$x_{ZIR}(t)$.Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At}X(0) = L^{-1}\left\{[sI - A]^{-1}X(0)\right\}$$

Fromtheaboverelation,wecanwritethestatetransitionmatrix**ϕ**(t)as

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

So,thezeroinputresponsecanbeobtainedbymultiplyingthestatetransitionmatrix **ϕ**(t) with the initial conditions matrix.

Propertiesofthestatetransitionmatrix

1. Ift=0,thenstatetransitionmatrixwillbeequal toanIdentitymatrix.

ϕ(0)=I

2. Inverseofstatetransitionmatrixwillbesameasthatofstatetransitionmatrixjustby replacing‘t’ by ‘-t’.

$$\phi^{-1}(t) = \phi(-t)$$

3. Ift=t1+t2,thenthecorrespondingstatetransitionmatrixisequaltothe multiplication of the two state transition matrices at t=t1t=t1 and t=t2t=t2.

ϕ(t1+t2)=**ϕ**(t1)**ϕ**(t2)

ControllabilityandObservability

Letusnowdiscusscontrollabilityandobservabilityofcontrolsystemonebyone.

Controllability

A control system is said to be **controllable** if the initial states of the control system are transferred(changed)tosomeotherdesiredstatesbyacontrolledinputinfinitedurationof time.

Wecancheckthecontrollabilityofa controlsystembyusing**Kalman’stest**.

4. WritethematrixQcinthefollowingform.

$$Q_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

5. Find the determinant of matrix QcQcand if it is not equal to zero, then the control system is controllable.

Observability

A control system is said to be**observable**if itis able to determine theinitial states of thecontrol system by observing the outputs in finite duration of time.

Wecanchecktheobservabilityofacontrolsystembyusing**Kalman'stest**.

6. WritethematrixQoinfollowingform.

$$Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T]$$

7. Find the determinant of matrix QoQoand if it is not equal to zero, then the control system is observable.

Example:

Letusverifythecontrollabilityandobservabilityofacontrolsystemwhichisrepresentedin the state space model as,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [0 \quad 1], D = [0] \quad and \quad n = 2$$

For $n = 2$, the matrix Q_c will be

$$Q_c = [B \quad AB]$$

We will get the product of matrices A and B as,

$$AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_c| = 1 \neq 0$$

Sincethedeterminantofmatrix Qc isnotequaltozero,thegivencontrolsystemis controllable.

Forn=2,thematrixQowillbe–

$$Q_o = [C^T \quad A^T C^T]$$

Here,

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We will get the product of matrices A^T and C^T as

$$A^T C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow |Q_o| = -1 \quad \neq 0$$

Since,thedeterminantofmatrix Qo isnotequaltozero,thegivencontrolsystemis observable. Therefore, the given control system is both controllable and observable.