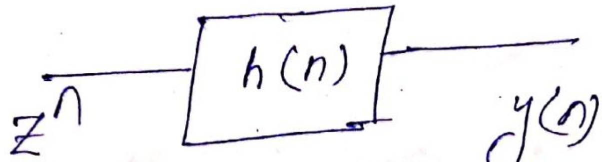
DSP (Z-Transform) 14-2-2020



$y(n) = z^n \cdot H(z)$

output

$$H(z) = Z[h(n)] = \sum_{n=-\infty}^{\infty} h(n) \cdot z^{-n}$$

If input is $x(n)$ Then it's Z-Transform is $Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

ROC = Region of convergence of $X(z)$ is set of all the values of z for which $X(z)$ has finite value.

$z = r \cdot e^{jw}$, r = Radius of the circle in z-domain

↳ If $x(n)$ is causal signal i.e. $x(n) = 0$ for $n<0$, Then The Z-Transform is

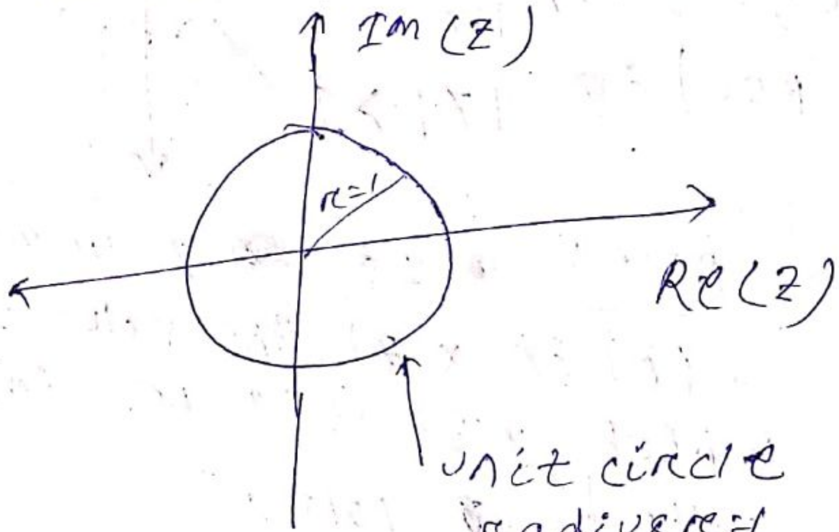
$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

Here it is one sided Z-Transform. It contains negative powers (-ve) of z in $X(z)$ expression.

→ If $x(n)$ is noncausal Discrete Time signal i.e. $x(n) = 0$ for $n > 0$ Then its Z-Transform is

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{-1} x(n) \cdot Z^{-n}$$

It contains positive powers (+ve) of Z in above expression. It is also one sided Z-Transform.



→ If $x(n) = u(n)$.

$$X(Z) = \sum_{n=0}^{\infty} u(n) \cdot Z^{-n}$$

$$= u(0) \cdot Z^{-0} + u(1) \cdot Z^{-1} + u(2) \cdot Z^{-2} + u(3) \cdot Z^{-3} + \dots$$

$$= 1 \cdot 1 + 1 \cdot Z^{-1} + 1 \cdot Z^{-2} + 1 \cdot Z^{-3} + \dots$$

$$= 1 + (Z^{-1})^1 + (Z^{-1})^2 + (Z^{-1})^3 + \dots$$

$$= 1 + x + x^2 + x^3 + \dots \quad [x = Z^{-1}]$$

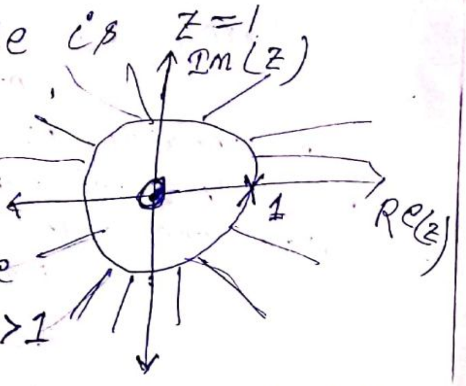
$$= \frac{1}{1-x} = \boxed{\frac{1}{1-Z^{-1}}} = \frac{1}{1-\frac{1}{Z}} = \frac{1}{\frac{Z-1}{Z}}$$

$$= \boxed{Z/Z-1}$$

$Z[u(n)] = \frac{z}{z-1}$

Here zero is $z = 0$

Here pole is $z = 1$



Here region of convergence (ROC) is $|z| > 1$

↳ **pole :-** If Z-Transform of $x(n)$ is $X(z)$. The value of z for which $X(z)$ will be infinite is called pole.

↳ **zero :-** The value of z for which $X(z)$ will be zero (0) is called zero.

$x(n) = \{1, \underset{\uparrow n=0}{2}, 3\}$ find $X(z)$.

$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=-1}^{1} x(n) \cdot z^{-n}$

$= x(-1) \cdot z^{-(-1)} + x(0) \cdot z^{-0} + x(1) \cdot z^{-1}$

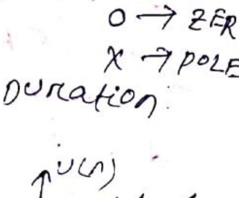
$= 1 \cdot z^{1} + 2 \cdot z + 3 \cdot z^{-1}$

Here ROC is all values of z except $z = 0$ and $z = \infty$.

$0 \rightarrow$ ZERO
$\times \rightarrow$ POLE

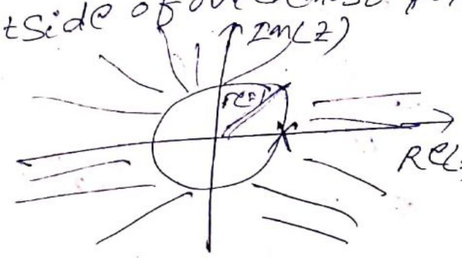
① Causal Infinite Duration Signal :-

Ex :- $u(n) = 1$; $n \geq 0$
$= 0$; for $n < 0$



$Z[u(n)] = \frac{z}{z-1}$

ROC :- outside of outeremost pole



① $Z[\delta(n)] = 1$

② If $Z[x(n)] = X(z)$ then
$Z[x(n-k)] = z^{-k} \cdot X(z)$,
$Z[x(n+k)] = z^{k} \cdot X(z)$

ROC = Entire Z-plane except $z = 0$

③ $Z[\delta(n-k)] = z^{-k} \cdot Z[\delta(n)]$
$= z^{-k} \cdot 1 = z^{-k}$

④ $Z[\delta(n+k)] = z^{k} \cdot Z[\delta(n)] = z^{k} \cdot 1 = z^{k}$
ROC :- Entire Z-plane except $z = \infty$.

Que:- Determine the Z-Transform and ROC of the signal $x(n) = a^n . u(n)$

Sol^n:- The given signal is causal and infinite duration.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot \underbrace{u(n)}_{1} \cdot Z^{-1}$$

$\boxed{u(n) = 1; \text{ for } n \geq 0}$

$$= \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n \qquad \text{put } az^{-1} = x$$

$$= x^0 + x^1 + x^2 + x^3 + x^4 + \ldots$$

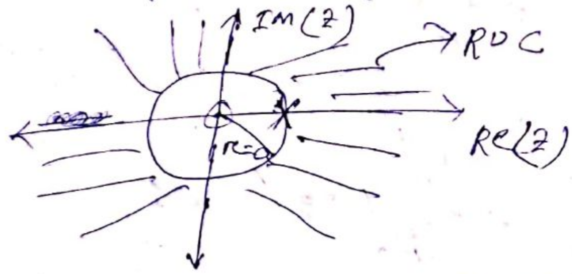
$$= \frac{1}{1-x} = \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}}$$

$$= \boxed{\frac{Z}{Z-a}}$$

for infinite duration causal signal, the ROC is outside of outer most pole.

Here one pole $z = a$



ZERO at $Z=0$, POLE at $Z=a$

ROC:- $|Z| > a$

Que:- find the Z-Transform and the ROC of the signal $x(n) = -b^n . u(-n-1)$

Sol^n:- $u(-n-1) = 1$, for $n \leq -1$

$= 0$, for $n \geq -1$

put $n=-1$, $\Rightarrow u[-(-1)-1] = u[1-1] = u[0] = 1$

put $n=-2$, $\Rightarrow u[-(-2)-1] = u[2-1] = u(1) = 1$

$$X(Z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -b^n \cdot \underbrace{u(-n-1)}_{1} \cdot z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n \cdot z^{-n} = -\sum_{n=1}^{\infty} \frac{1}{b^n \cdot z^{-n}}$$

$$= -\sum_{n=1}^{\infty} b^{-n}.z^{n} = -\sum_{n=1}^{\infty}(b^{-1}z)^{n}$$

$$= -\left[\sum_{n=1}^{\infty}(b^{-1}z)^{n} + (b^{-1}z)^{0} - (b^{-1}z)^{0}\right]$$

$$= -\left[\sum_{n=0}^{\infty}(\underbrace{b^{-1}z}_{x})^{n} - 1\right]$$

$$= -\left[\frac{1}{1-x} - 1\right]$$

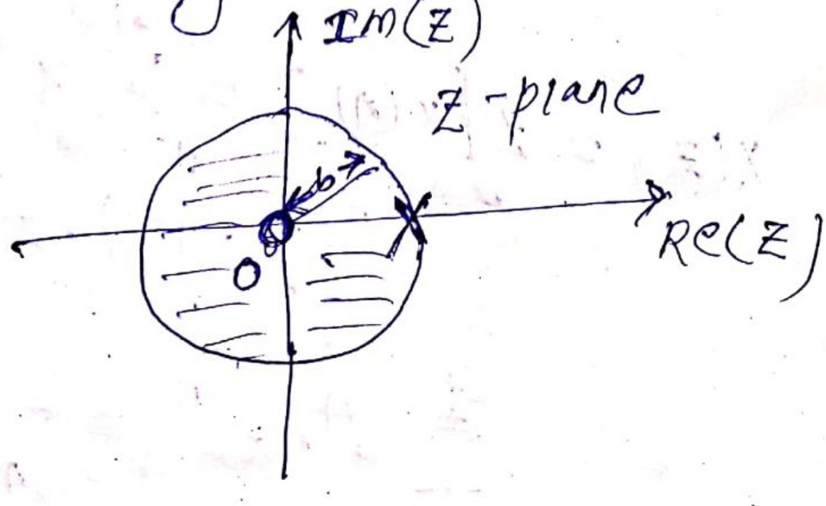
$$= -1\left[\frac{1}{1-b^{-1}z} - 1\right] = -\left[\frac{1}{1-\frac{z}{b}} - 1\right]$$

$$= -\left[\frac{1}{\frac{b-z}{b}} - 1\right] = -\left[\frac{b}{b-z} - 1\right]$$

$$= 1 - \frac{b}{b-z} = \frac{b-z-b}{b-z} = \frac{-z}{b-z}$$

ROC :- $b-z>1 \Rightarrow b>z \Rightarrow \boxed{z<b}$

The ROC is now the interciore of a circle having radius b



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Digital signal processing

Z-TRANSFORM :-

$$Z[a^n \cdot u(n)] = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \text{ROC}: |Z| > a$$

$$Z[-a^n \cdot u(-n-1)] = \frac{1}{1-az^{-1}}, \text{Roc}: |Z| < a$$

$$= \frac{z}{z-a}$$

ROC of Two sided sequence :-

↳ The ROC of a causal signal is exterior of a circle of radius r. The ROC of of a anticausal signal is interior of a circle of radius r.

Let us consider a Two sided sequence is

$$x(n) = a^n \cdot u(n) + b^n \cdot u(-n-1)$$

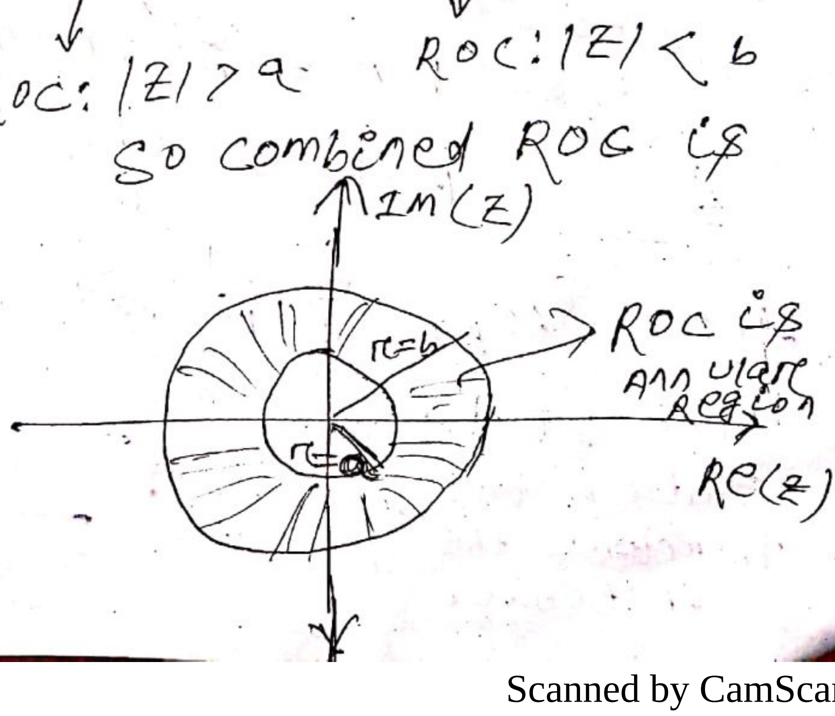
$$X(Z) = \frac{1}{1-aZ^{-1}} - \frac{1}{1-bZ^{-1}}$$

$\frac{1}{1-aZ^{-1}}$ ↓ ROC: $|Z| > a$

$\frac{1}{1-bZ^{-1}}$ ↓ ROC: $|Z| < b$

So combined ROC is

$$a < |Z| < b$$



Stability and ROC :-

$y(n) = x(n) * h(n)$ $\quad x(n) \rightarrow \boxed{\text{system } h(n)} \rightarrow y(n)$

Let $h(n)$ is the impulse Response of a causal (or) Non causal Linear Time invariant system and $H(Z)$ be the Z-Transform of $h(n)$. Then stability of the system can be found from ROC using the following Theorem.

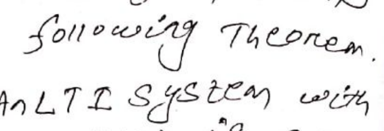
Theorem :- An LTI System with the system function $H(Z)$ is BIBO stable (Bounded input Bounded output) if and only if the ROC for $H(Z)$ contains the unit circle.

Que:- Find the stability of the system whose impulse Response is $h(n) = (2)^n \cdot u(n)$

Sol:- $H(Z) = Z[(2)^n \cdot u(n)]$

$$= \frac{1}{1-2\cdot Z^{-1}} = \frac{Z}{Z-2}; \text{ ROC is } |Z| > 2$$

Here, The ROC is $|Z|>2$. It does not contain the unit circle. Therefore the system is unstable.



Properties of Z-Transform:-

① **Linearity:-** If $Z[x_1(n)] = X_1(Z)$; ROC $= R_1$

$Z[x_2(n)] = X_2(Z)$; ROC $= R_2$

Then $x(n) = a\cdot x_1(n) + b\cdot x_2(n)$ having Z-Transform is

$$Z[x(n)] = X(Z) = a\cdot Z[x_1(n)] + b\cdot Z[x_2(n)]$$
$$= a\cdot X_1(Z) + b\cdot X_2(Z)$$

ROC is : $R_1 \cap R_2$

a, b are constants

Que:- The signal is given by $x(n) = [2(3^n) - 3(4^n)]u(n)$. Determine Z-Transform using Linearity property.

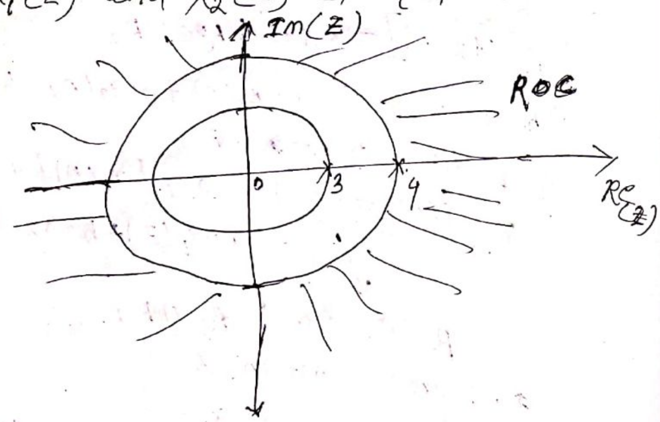
Sol:- $x(n) = 2(3)^n u(n) - 3(4)^n u(n)$

$$Z[x(n)] = X(Z) = 2 Z[(3)^n u(n)] - 3\cdot Z[(4)^n u(n)]$$
$$= 2\cdot\frac{1}{1-3\cdot Z^{-1}} - 3\cdot\frac{1}{1-4\cdot Z^{-1}}$$
$$= 2\cdot\frac{Z}{Z-3} - 3\cdot\frac{Z}{Z-4}$$

ROC(R_1): $|Z|>3$ $\quad$ ROC(R_2): $|Z|>4$

$R_1 \cap R_2$ is ROC $|Z| > 4$

The intersection of ROC of $X_1(Z)$ and $X_2(Z)$ is $|Z| > 4$.



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Que: consider the Z-Transform

$$X(Z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)},$$

ROC: $|Z| > \frac{1}{2}$, $|Z| > \frac{1}{4}$.

Determine $x(n)$.

Sol^n:

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}}$$

$$A = \lim_{z^{-1}\to 2} \left(1-\frac{1}{2}z^{-1}\right)\cdot X(z)$$

$$= \lim_{z^{-1}\to 2} \cancel{\left(1-\frac{1}{2}z^{-1}\right)} \cdot \frac{1}{\cancel{\left(1-\frac{1}{2}z^{-1}\right)}\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{1}{1-\frac{1}{4}\times 2} = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = \boxed{2}$$

$$B = \lim_{z^{-1}\to 4} \left(1-\frac{1}{4}z^{-1}\right)\cdot X(z)$$

$$= \lim_{z^{-1}\to 4} \cancel{\left(1-\frac{1}{4}z^{-1}\right)} \cdot \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\cancel{\left(1-\frac{1}{4}z^{-1}\right)}}$$

$$= \lim_{z^{-1}\to 4} \frac{1}{1-\frac{1}{2}\cdot z^{-1}} = \frac{1}{1-\frac{1}{2}\times 4} = \frac{1}{1-2}$$

$$= \frac{1}{(-1)} = \boxed{-1}$$

$$X(Z) = 2\cdot\frac{1}{1-\frac{1}{2}z^{-1}} - 1\cdot\frac{1}{1-\frac{1}{4}\cdot z^{-1}}$$

$$Z^{-1}[X(Z)] = x(n)$$

$$= 2\cdot Z^{-1}\left[\frac{1}{1-\frac{1}{2}\cdot z^{-1}}\right] - 1\cdot Z^{-1}\left[\frac{1}{1-\frac{1}{4}z^{-1}}\right]$$

$$= 2\cdot\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n\cdot u(n)$$

$$Z^{-1}\left[a^n\cdot u(n)\right] = Z^{-1}\left[-a^n\cdot u(-n-1)\right]$$

$$= \frac{1}{1-a\cdot z^{-1}}$$

$Z[x(n-k)] = z^{-k} \cdot X(z)$

proof:-

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

put $n-k = l$

$$Z[x(n-k)] = Z[x(l)] = \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-l}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-(n-k)}$$

24-2-2020

Digital signal processing

Que: $x(n) = \cos\omega n \cdot u(n)$, find Z-Transform.

Soln: $x(n) = \cos\omega n \cdot u(n)$

$$= \frac{1}{2}\left[e^{j\omega n} + e^{-j\omega n}\right] u(n)$$

$$= \frac{1}{2} \cdot e^{j\omega n} \cdot u(n) + \frac{1}{2} \cdot e^{-j\omega n} \cdot u(n)$$

$$= \frac{1}{2} x_1(n) + \frac{1}{2} x_2(n)$$

$$X(Z) = \frac{1}{2} \cdot X_1(Z) + X_2(Z) \cdot \frac{1}{2}$$

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\omega n} \cdot u(n) \cdot Z^{-n}$$

$\boxed{u(n) = 1;\ \text{for } n \geq 0}$

$$= \sum_{n=0}^{\infty} e^{j\omega n} \cdot 1 \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(e^{j\omega} \cdot Z^{-1}\right)}_{a}^{n}$$

$$\left[1 + a + a^2 + a^3 + \cdots \infty = \frac{1}{1-a}\right.$$

$$= \frac{1}{1 - e^{j\omega} \cdot Z^{-1}}$$

$$X_2(Z) = \sum_{n=0}^{\infty} e^{-j\omega n} \cdot Z^{-n} = \sum_{n=0}^{\infty} \left(e^{-j\omega} \cdot Z^{-1}\right)^n$$

$$= \boxed{\frac{1}{1 - e^{-j\omega} \cdot Z^{-1}}}$$

$$X(z) = \frac{1}{2}\cdot X_1(z) + \frac{1}{2}\cdot X_2(z)$$

$$= \frac{1}{2}\cdot\left[\frac{1}{1-e^{jw}\cdot z^{-1}} + \frac{1}{1-e^{-jw}\cdot z^{-1}}\right]$$

$$= \frac{1}{2}\cdot\left[\frac{1-e^{-jw}\cdot z^{-1} + 1 - e^{jw}\cdot z^{-1}}{(1-e^{jw}\cdot z^{-1})(1-e^{-jw}\cdot z^{-1})}\right]$$

$$= \frac{1}{2}\cdot\left[\frac{2 - e^{jw} z^{-1} - e^{-jw} z^{-1}}{1-e^{-jw}\cdot z^{-1} - e^{jw}\cdot z^{-1} + e^{jw}\cdot z^{-1}\cdot e^{-jw}\cdot z^{-1}}\right]$$

$$= \frac{1}{2}\cdot 2\left[\frac{1 - \frac{z^{-1}(e^{jw}+e^{-jw})}{2}}{1 - z^{-1}(e^{jw}+e^{-jw}) + z^{-2}}\right]$$

$e^{j0}\cdot e^{j0} = 1$

$$= \left[\frac{1 - z^{-1}\cdot\cos w}{1 - z^{-1}\cdot 2\cdot\frac{(e^{jw}+e^{-jw})}{2} + z^{-2}}\right]$$

$z^{-1}\cdot z^{-1} = z^{-2}$

$$= \left[\frac{1-z^{-1}\cdot\cos w}{1 - z^{-1}\cdot 2\cdot\cos w + z^{-2}}\right]$$

$\cos w = \frac{e^{jw}+e^{-jw}}{2}$

$$= \left[\frac{1-z^{-1}\cdot\cos w}{1-2z^{-1}\cdot\cos w + z^{-2}}\right]$$

$$= Z\left[\cos wn\cdot u(n)\right]$$

TIME SHIFTING

If $Z[x(n)] = X(z)$ then

$$Z[x(n-k)] = z^{-k} \cdot X(z)$$

$$Z[x(n+k)] = z^{k} \cdot X(z)$$

Que:- $x_1(n) = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$

$$X(z) = \sum_{n=0}^{4} x_1(n) \cdot z^{-n}$$

$$= x_1(0) \cdot z^{-0} + x_1(1) \cdot z^{-1} + x_1(2) \cdot z^{-2} + x_1(3) \cdot z^{-3} + x_1(4) \cdot z^{-4}$$

$$= 1 \cdot z^{-0} + 2 \cdot z^{-1} + 3 \cdot z^{-2} + 4 \cdot z^{-3} + 5 \cdot z^{-4}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4 \cdot z^{-3} + 5 \cdot z^{-4}$$

ROC: Entire z-plane except $z = 0$

$$x_2(n) = \{ 1, 2, \underset{\substack{\uparrow \\ n=0}}{3}, 4, 5 \}$$

$$= x_1(n+2)$$

$$Z[x_1(n+2)] = z^{2} \cdot X_1(z)$$

$$X_2(z) = Z[x_1(n+2)]$$

$$= z^{2} \cdot [1 + 2z^{-1} + 3z^{-2} + 4 \cdot z^{-3} + 5 \cdot z^{-4}]$$

$$= z^{2} + 2 \cdot z^{1} + 3 + 4 \cdot z^{-1} + 5 \cdot z^{-2}$$

ROC:- Entire z-plane except $z = 0$ and $z = \infty$.

$$x_3(n) = \{0, 0, 1, 2, 3, 4, 5\}$$
$$\uparrow$$

$$= x_1(n-2)$$

$$X_3(z) = Z[x_3(n)] = Z[x_1(n-2)]$$

$$= z^{-2} \cdot X_1(z)$$

$$= z^{-2} \cdot [1 + 2z^{-1} + 3z^{-2} + 4 \cdot z^{-3} + 5 \cdot z^{-4}]$$

$$= z^{-2} + 2 \cdot z^{-3} + 3z^{-4} + 4 \cdot z^{-5} + 5 \cdot z^{-6}$$

ROC :- Entire z-plane except $z = 0$.

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TIME REVERSAL :-

If $x(n) \xleftrightarrow{Z} X(z)$, ROC: $r_1 < |z| < r_2$

Then

$x(-n) \xleftrightarrow{Z} X(z^{-1})$, ROC :

$$\frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Que :- $x(n) = a^{-n} \cdot u(-n)$, find Z-Transform by using Time Reversal property.

Soln :- $Z[\underbrace{a^n \cdot u(n)}_{x_1(n)}] = \frac{1}{1 - a \cdot z^{-1}}$

$= X_1(z)$ ROC: $|z| > a$

$Z[x_1(-n)] = X_1(z^{-1})$

$$Z[a^{-n}.u(-n)] = \frac{1}{1-a.(z^{-1})^{-1}}$$

$$= \boxed{\frac{1}{1-a.z}}$$

$\downarrow$ ROC: $|z| < \frac{1}{a}$

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SCALING IN Z-DOMAIN:- $\rightarrow$ ROC: $r_1 < |z| < r_2$

If $Z[x(n)] = X(z)$ Then

$$Z[a^n.x(n)] = X(a^{-1}z) \rightarrow \text{ROC}: |a|r_1 < |z| < |a|r_2$$

---x---

Que:- If $x(n) = a^n.\sin w_0 n.u(n)$

Then find its Z-Transform.

Sol^n:- $$Z[\sin w_0 n.u(n)] = \frac{z^{-1}.\sin w_0}{1-2z^{-1}.\cos w_0 + z^{-2}}$$

$$Z[a^n.\sin w_0 n.u(n)]$$

$$= \frac{(a^{-1}z)^{-1}.\sin w_0}{1-2(a^{-1}z)^{-1}.\cos w_0 + (a^{-1}z)^{-2}}$$

$$= \frac{a.z^{-1}.\sin w_0}{1-2a.z^{-1}.\cos w_0 + a^2.z^{-2}}$$

$$\sin w_0 n = \frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}$$

$$Z\left[\sin\omega_0 n \cdot u(n)\right] = \sum_{n=-\infty}^{\infty} \sin\omega_0 n \cdot \underbrace{u(n)}_{1} \cdot z^{-n}$$

$u(n) = 1; n \geq 0$
$= 0; n < 0$

$$= \sum_{n=0}^{\infty}\left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right) \cdot z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega_0 n} \cdot z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} \cdot z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \underbrace{(e^{j\omega_0} \cdot z^{-1})}_{a}{}^{n} - \sum_{n=0}^{\infty} \underbrace{(e^{-j\omega_0} \cdot z^{-1})}_{b}{}^{n}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0} \cdot z^{-1}} - \frac{1}{1-e^{-j\omega_0} \cdot z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\omega_0} \cdot z^{-1} - (1-e^{j\omega_0} \cdot z^{-1})}{(1-e^{j\omega} \cdot z^{-1})(1-e^{-j\omega} \cdot z^{-1})}\right]$$

$$= \frac{1}{2j}\left[\frac{1-e^{-j\omega} \cdot z^{-1} - 1 + e^{j\omega} \cdot z^{-1}}{1-e^{-j\omega} \cdot z^{-1} - e^{j\omega} \cdot z^{-1} + e^{j\omega} \cdot z^{-1} \cdot e^{-j\omega} \cdot z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{z^{-1}e^{j\omega} - z^{-1}e^{-j\omega}}{1-z^{-1}(e^{j\omega}+e^{-j\omega}) + z^{-2}}\right]$$

$$= \frac{z^{-1}\left(\frac{e^{j\omega}-e^{-j\omega}}{2j}\right)}{1 - z^{-1} \cdot 2 \cdot \left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) + z^{-2}}$$

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$$= \frac{z^{-1} \cdot \sin\omega_0}{1 - z^{-1} \cdot 2 \cdot \cos\omega_0 + z^{-2}}$$

$$= \boxed{\frac{z^{-1} \cdot \sin\omega_0}{1 - 2z^{-1} \cdot \cos\omega_0 + z^{-2}}}$$

——— x ———

Differentiation in Z-Domain :-

If $Z[x(n)] = X(Z)$ Then

$$Z[n \cdot x(n)] = -Z \cdot \frac{dX(Z)}{dZ}$$

Que:- Using differentiation property determine the Z-Transform of the signals $x(n) = n \cdot u(n)$

Sol^n:- $Z[a^n \cdot u(n)] = \frac{1}{1 - aZ^{-1}}$

$$Z[(1)^n \cdot u(n)] = \frac{1}{1 - 1 \cdot Z^{-1}}$$

$$\Rightarrow Z[u(n)] = \frac{1}{1 - Z^{-1}} = \frac{1}{1 - \frac{1}{Z}}$$

$$= \frac{1}{\frac{Z-1}{Z}} = \boxed{\frac{Z}{Z-1}}$$

$$Z[n \cdot u(n)] = -Z \cdot \frac{d}{dZ}\left(\frac{Z}{Z-1}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'.v - u.v'}{v^2}$$

$$u' = \frac{du}{dx}, \quad v' = \frac{dv}{dx}$$

$$= -Z\left[\frac{\frac{dZ}{dZ}.(Z-1) - Z.\frac{d(Z-1)}{dZ}}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{(Z-1) - Z.1}{(Z-1)^2}\right]$$

$$= -Z.\left[\frac{\cancel{Z}-1-\cancel{Z}}{(Z-1)^2}\right]$$

$$= \frac{Z}{(Z-1)^2}$$

xxx ——— xxx

Que:- Deteremine the signal $x(n)$ where Z-Transform is given by

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a|$$

Sol^n:-
$$\frac{dX(Z)}{dZ} = \frac{d.\log(1+aZ^{-1})}{d(1+aZ^{-1})} . \frac{d(1+aZ^{-1})}{dZ}$$

$$= \frac{1}{(1+aZ^{-1})} . a . \frac{dZ^{-1}}{dZ}$$

$$= a . \frac{1}{(1+aZ^{-1})} . (-1) . Z^{-1-1}$$

$$= -a \cdot \frac{1}{(1+az^{-1})} \cdot z^{-2}$$

$$-z \cdot \frac{dX(z)}{dz} = \frac{-z \cdot (-a) \cdot z^{-2}}{(1+az^{-1})}$$

$$= \frac{z^{-1} \cdot a}{1+az^{-1}}$$

$$= a \cdot z^{-1} \cdot \frac{1}{1-(-a)z^{-1}}$$

Take Inverse Z-Transform on both sides

$$a^n \cdot u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$$

$$(-a)^n u(n) \xleftrightarrow{Z} \frac{1}{1-(-a)z^{-1}}$$

$$(-a)^{n-1} u(n-1) \xleftrightarrow{Z} \frac{z^{-1}}{1+az^{-1}}$$

$$n \cdot x(n) = a \cdot (-a)^{n-1} \cdot u(n-1)$$

$$\Rightarrow x(n) = \frac{1}{n} \cdot a^1 \cdot (-1)^{n-1} \cdot (a)^{n-1} \cdot u(n-1)$$

$$= \frac{1}{n} \cdot a^n \cdot (-1)^{n-1} \cdot u(n-1)$$

Convolution property :-

If Two signals $x_1(n)$ and $x_2(n)$ and we will make convolution

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

$$Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$$

That is convolution of Two signals in Time Domain is product of Two signals in Z-Domain

The Region of convergence (ROC) of $X(Z)$ is the intersection of $X_1(Z)$ and $X_2(Z)$.

Que:- Find the convolution of signal $x_1(n) = a^n \cdot u(n)$, $x_2(n) = u(n)$ using Z-Transform.

Sol^n:- $X_1(Z) = Z[x_1(n)] = Z[a^n \cdot u(n)]$

$$= \frac{1}{1-a \cdot z^{-1}},$$

ROC: $|Z| > |a|$

$$X_2(Z) = Z[u(n)] = \frac{1}{1-Z^{-1}}, \text{ ROC: } |Z| > |1|$$

$$x(n) = x_1(n) * x_2(n)$$

$$Z[x(n)] = X(Z) = Z[x_1(n) * x_2(n)]$$

$$= X_1(Z) \cdot X_2(Z)$$

$$\Rightarrow X(Z) = \frac{1}{1-aZ^{-1}} \cdot \frac{1}{1-Z^{-1}},$$ By making partial fraction

$$= \frac{A}{1-aZ^{-1}} + \frac{B}{1-Z^{-1}}$$

$$A = \lim_{Z^{-1} \to \frac{1}{a}} (1-aZ^{-1}) \cdot X(Z)$$

$$= \lim_{Z^{-1} \to \frac{1}{a}} \cancel{(1-aZ^{-1})} \cdot \frac{1}{\cancel{(1-aZ^{-1})}(1-Z^{-1})}$$

$$= \frac{1}{1-\frac{1}{a}} = \frac{1}{\frac{a-1}{a}} = \frac{a}{a-1}$$

$$B = \lim_{Z^{-1}\to 1} (1-Z^{-1}) \cdot X(Z)$$

$$= \lim_{Z^{-1}\to 1} \cancel{(1-Z^{-1})} \cdot \frac{1}{(1-aZ^{-1})\cancel{(1-Z^{-1})}}$$

$$= \frac{1}{1-a\times 1} = \frac{1}{1-a}$$

$$X(Z) = \frac{a/a-1}{1-aZ^{-1}} + \frac{1/1-a}{1-aZ^{-1}}$$

$$= \frac{-\frac{a}{1-a}}{1-aZ^{-1}} + \frac{1/1-a}{1-Z^{-1}}$$

$$= \frac{1}{1-a}\left[\frac{1}{1-Z^{-1}} - \frac{a}{1-aZ^{-1}}\right]$$

$=$

Making inverse Z-Transform

$$Z^{-1}[X(Z)] =$$

$$= x(n)$$

$$= \frac{1}{1-a} \cdot [u(n) - a \cdot a^n \cdot u(n)]$$

$$\Rightarrow x(n) = \frac{1}{1-a}[u(n) - a^{n+1} \cdot u(n)]$$

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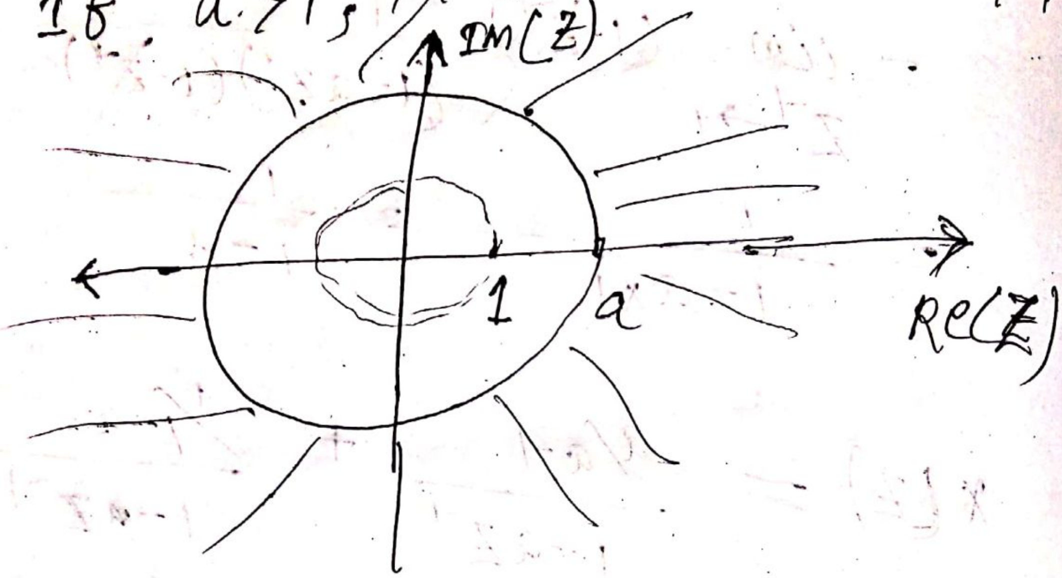
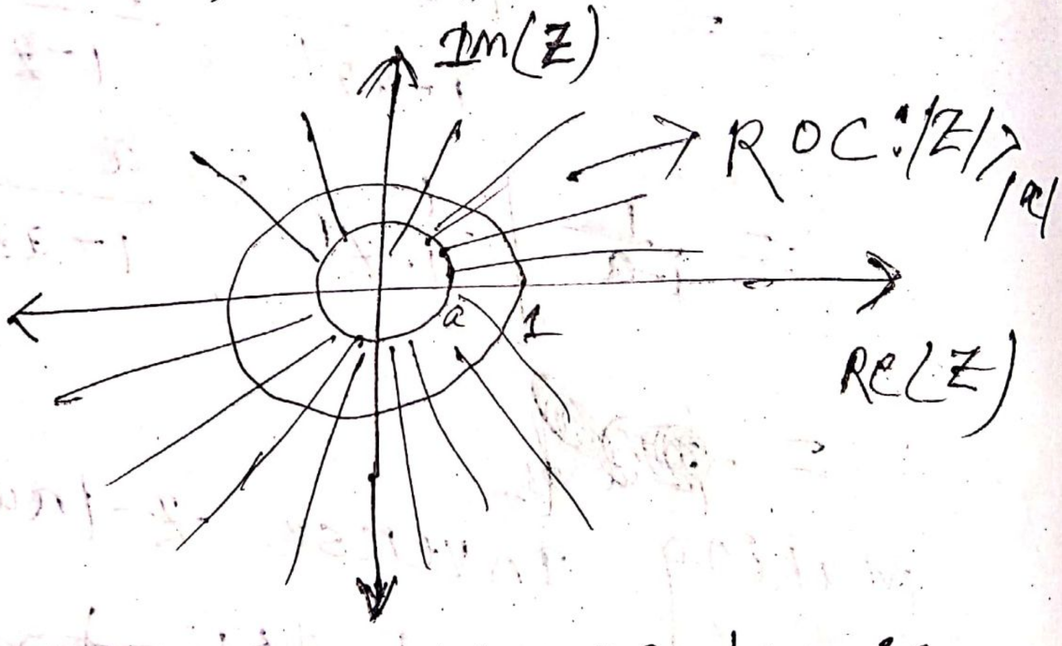
ROC : $R_1 \cap R_2$

$|z| > |a|$ $\quad$ $|z| > 1$

If $a > 1$, Then ROC $R : |z| > |a|$



If $a < 1$, Then ROC $R : |z| > |a|$



The convolution property is most powerful property of the Z-Transform.

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Initial Value Theorem:-

If $x(n) = 0$ for $n < 0$, Then

$$x(0) = \lim_{z \to \infty} X(z)$$

proof:-

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\Rightarrow X(z) = x(0) \cdot z^{-0} + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + \cdots$$

$$= x(0) \cdot 1 + x(1) \cdot \frac{1}{z} + x(2) \frac{1}{z^2} + \cdots$$

$$\lim_{z \to \infty} X(z) = x(0) + \frac{x(1)}{\infty} + x(2) \frac{1}{\infty} + \cdots$$

$$= x(0) + 0 + 0 + 0 \cdots$$

$$= x(0)$$

Correlation of Two sequences:-

If Two signals $x_1(n)$ and $x_2(n)$ Then Their correlation is

$$r_{x_1 x_2}(n) = x_1(n) * x_2(-n)$$

$$\updownarrow z \qquad\qquad \updownarrow z \rightarrow \text{Z-Transform}$$

$$R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

The ROC of $R_{x_1 x_2}$ is the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$.

Que:- Find the Z-Transform and ROC of the signal sequence.

$$x(n) = [4(2^n) - 5(3^n)]u(n)$$

Sol^n:- $x(n) = 4.(2)^n.u(n) - 5(3)^n.u(n)$

$= 4.x_1(n) - 5.x_2(n)$

$x_1(n) = (2)^n.u(n)$

$X_1(Z) = Z[(2)^n.u(n)] = \frac{1}{1-2Z^{-1}}$

~~$X_1(Z) = Z[$~~ ROC $R_1 : |Z| > |2|$

$X_2(Z) = Z[(3)^n.u(n)] = \frac{1}{1-3Z^{-1}}$

ROC $R_2 : |Z| > |3|$

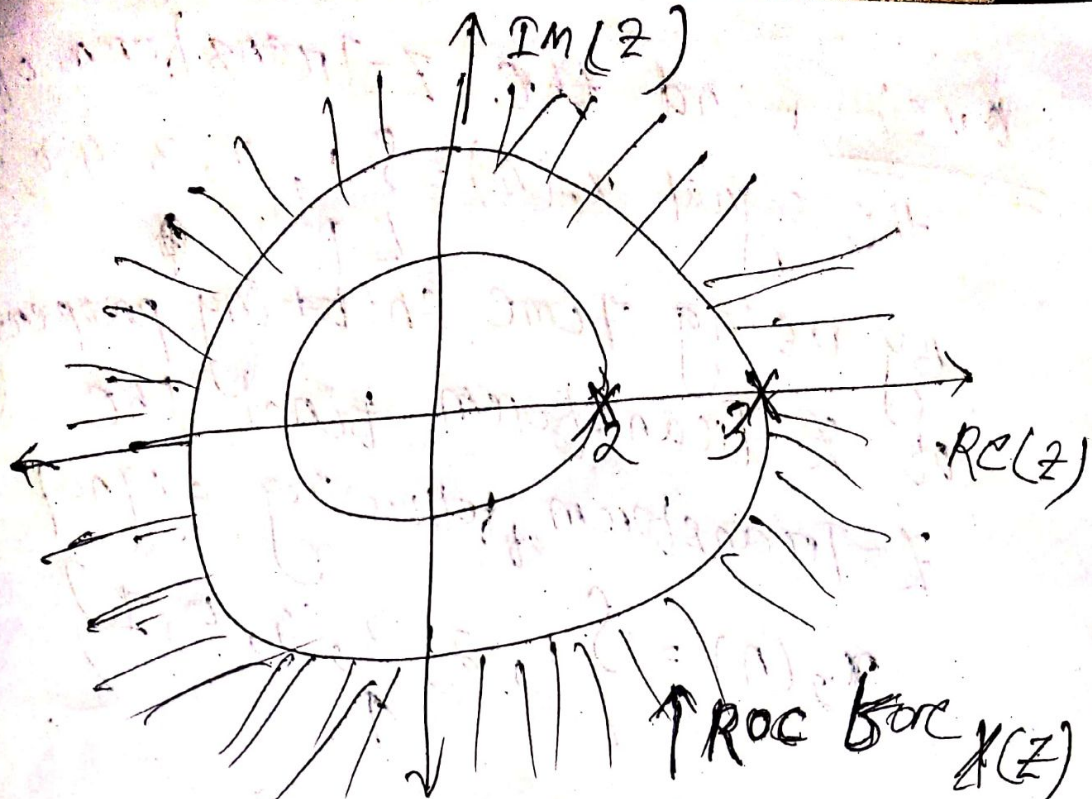
ROC $R : R_1 \cap R_2$

$|Z| > |2|, \quad |Z| > |3|$

$\boxed{\Rightarrow \text{ROC } R : |Z| > |3| \text{ for } X(Z)}$

$X(Z) = Z[x(n)]$

$= 4.\frac{1}{1-2.Z^{-1}} - 5.\frac{1}{1-3Z^{-1}}$



ROC for $X(z)$

is $R: R_1 \cap R_2$

$: |z| > |3|$

Que:- Find the z-Transform of sequence $x(n) = u(-n)$

Sol:- we know $z[x(n)] = X(z)$

and $z[x(-n)] = X(z^{-1})$

Similary we know the z-Transform of $u(n)$ is

$$z[u(n)] = \frac{1}{1-z^{-1}}$$

and $$z[u(-n)] = \frac{1}{1-(z^{-1})^{-1}}$$

$$= \frac{1}{1+z} \; ; \; ROC: |z| > 1$$

Que:- Find the Z-Transform of the signal $x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 0, 1\}$.

By using Time shifting property of Z-Transform find the Z-Transform of following signal

$x_2(n) = \{1, 2, \underset{\uparrow}{3}, 4, 0, 1\}$

Soln:- $x_2(n) = x_1(n+2)$

$$X_2(Z) = Z[x_1(n+2)] = Z^2 \cdot X_1(Z) \quad \text{--- (1)}$$

$$X_1(Z) = \sum_{n=0}^{5} x(n) \cdot Z^{-n}$$

$$= x(0) \cdot Z^{-0} + x(1) \cdot Z^{-1} + x(2) \cdot Z^{-2} + x(3) \cdot Z^{-3} + x(4) \cdot Z^{-4} + x(5) \cdot Z^{-5}$$

$$= 1 \cdot Z^{-0} + 2 \cdot Z^{-1} + 3 \cdot Z^{-2} + 4 \cdot Z^{-3} + 0 \cdot Z^{-4} + 1 \cdot Z^{-5}$$

$$= 1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + Z^{-5}$$

From Eqn (1) ROC: $R_1 \rightarrow$ Entire Z-plane except $Z = 0$

$$X_2(Z) = Z^2 \cdot X_1(Z)$$

$$= Z^2 \cdot [1 + 2Z^{-1} + 3Z^{-2} + 4 \cdot Z^{-3} + Z^{-5}]$$

$= z^{2} + 2\cdot z^{1} + 3\cdot z^{0} + 4\cdot z^{-1} + z^{-3}$

ROC : RZ → Entire z-plane except $z = 0$ and $z = \infty$.

Que :- Find the convolution of following two sequences, by using Z - transform Property.

$x_1(n) = \{\underset{\uparrow}{1}, -2, 1\}$, $x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

A) $x_1(z) = Z[x_1(n)] = \sum_{n=0}^{2} x_1(n).z^{-n}$

$= x_1(0).z^{-0} + x_1(1).z^{-1} + x_1(2).z^{-2}$

$= 1.z^{-0} + (-2).z^{-1} + 1.z^{-2}$

$= 1 - 2.z^{-1} + z^{-2}$

$x_2(z) = Z[x_2(n)] = \sum_{n=0}^{2} x_2(n) z^{-n}$

$= x_2(0).z^{-0} + x_2(1).z^{-1} + x_2(2).z^{-2}$

$= 1.z^{-0} + 1.z^{-1} + 1.z^{-2} = 1 + z^{-1} + z^{-2}$

Take $x(n) = x_1(n) * x_2(n)$

By making Z - transform on both sides.

$\Rightarrow Z[x(n)] = Z[x_1(n) * x_2(n)] = x_1(z).x_2(z)$

$\Rightarrow x(z) = (1 - 2z^{-1} + z^{-2}).(1 + z^{-1} + z^{-2})$

$\Rightarrow x(z) = 1 + z^{-1} + z^{-2} - 2z^{-1} - 2z^{-2} - 2z^{-3} + z^{-2} + z^{-3} + z^{-4}$

$\Rightarrow x(z) = 1 - z^{-1} + 0.z^{-2} - z^{-3} + z^{-4} \longrightarrow (1)$

By taking inverse Z transform to above equation.

$\Rightarrow x(n) = \{1, -1, 0, -1, 1\}$.

Que Find the Z - transform of the sequence

$x(n) = \left(\frac{1}{3}\right)^{n-1}.u(n-1)$.

A) We know $Z[x(n-k)] = z^{-k}.x(z)$

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$\because z\,(x(n+k)] = z^{k}.x(z)$

$$z\left[\left(\frac{1}{3}\right)^{n}.u(n)\right] = \frac{1}{1-\frac{1}{3}.z^{-1}}$$

$$\therefore z[x(n)] = X(z) = z\left[\left(\frac{1}{3}\right)^{n-1}.u(n-1)\right]$$

$$= z^{-1}.\frac{1}{1-\frac{1}{3}.z^{-1}}$$

Que :- Find the Z-transform of the sequence.
$x(n) = n.a^{n}.u(n)$

A) We know if $x(n) \overset{z}{\longleftrightarrow} X(z)$

Then $n.x(n) \overset{z}{\longleftrightarrow} -z.\frac{dX(z)}{dz}$

Here,

$x(n) = n.s(n)$, where $s(n) = a^{n}.u(n)$

$S(z) = z[s(n)] = z[a^{n}.u(n)]$

$$= \frac{1}{1-a.z^{-1}} = \frac{1}{1-a.\frac{1}{z}} = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

Now making Z transform of $x(n)$

$z[x(n)] = X(z) = z[n.s(n)]$

$$= -z.\frac{dS(z)}{dz} \longrightarrow (1)$$

$$\therefore \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-u.v'}{v^{2}}$$

$u' = \frac{du}{dx}; v' = \frac{dv}{dx}$

$$\Rightarrow X(z) = -z.\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$\Rightarrow X(z) = -z\left[\frac{\frac{dz}{dz}(z-a) - z.\frac{d(z-a)}{dz}}{(z-a)^{2}}\right]$$

$$\Rightarrow X(z) = -z\left[\frac{1(z-a)-z.1}{(z-a)^{2}}\right]$$

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$$= -z\left[\frac{z-a-z}{(z-a)^2}\right] \Rightarrow X(z) = \frac{a \cdot z}{(z-a)^2}$$

Que :- Find the System function & impulse response of the system described by the difference eqn.

$$y(n) = \frac{1}{5} y(n-1) + x(n)$$

A) given, system is $y(n) = \frac{1}{5} y(n-1) + x(n)$

Take z transform on both sides.

$$\Rightarrow Z[y(n)] = \frac{1}{5} Z[y(n-1)] + Z[x(n)]$$

$$\Rightarrow Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$$

$$\Rightarrow Y(z) - \frac{1}{5} z^{-1} Y(z) = X(z)$$

$$\Rightarrow Y(z) \cdot \left[1 - \frac{1}{5} z^{-1}\right] = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5} z^{-1}}$$

$$\Rightarrow H(z) = \text{system function} = \frac{1}{1 - \frac{1}{5} z^{-1}}$$

we will find inverse z-transform is

Impulse response $= h(n) = \left(\frac{1}{5}\right)^n \cdot u(n)$

Ques :- $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

A) Here $x(n)$ is i/p & $y(n)$ is o/p

The given system is

$$y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$$

Take z-transform on both sides we will get,

$$\Rightarrow Y(z) = X(z) + 2z^{-1} X(z) - 4 \cdot z^{-2} X(z) + z^{-3} X(z)$$

$$\Rightarrow Y(z) = X(z)\left[1 + 2 \cdot z^{-1} - 4 \cdot z^{-2} + z^{-3}\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \cancel{X(z)} \left(1 + 2 \cdot z^{-1} - 4 \cdot z^{-2} + z^{-3}\right] = H(z) = \text{System function}$$

By making inverse z transform on both sides we will get

$\Rightarrow h(n) = \{ \underset{\uparrow}{1}, 2, -4, 1 \}$ = Impulse response of the system.

Que Find the pole-zero plot for the system described by the difference eqn

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

$\Rightarrow$ Given, difference eqn is

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$$

Taking z-transform on both sides, we get

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

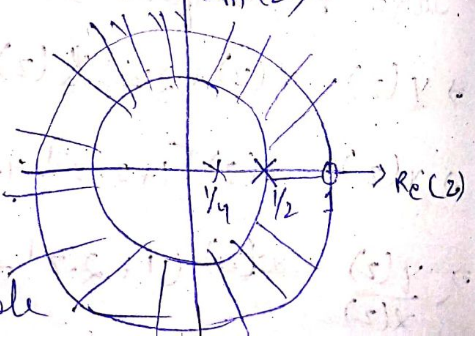
$$\Rightarrow Y(z) \cdot \left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2} \right] = X(z) \cdot \left[1 - z^{-1} \right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{2} z^{-1} + \frac{1}{8} z^{-2}} = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right) + \frac{1}{8} z^{-2} - \frac{1}{4} z^{-1}}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right) + \frac{1}{4} z^{-1} \left(\frac{1}{2} z^{-1} - 1\right)} = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right) - \frac{1}{4} z^{-1} \left(1 - \frac{1}{2} z^{-1}\right)}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)}$$



ROC: $|z| > \frac{1}{2}$

Outerside of outermost pole

of system function
Here, ROC includes the unit circle; Hence the system is stable.

> Here ROC of system function cannot contain any pole.

Que :- Find the Z-transform & ROC of given signal
$x(n) = \frac{1}{2}\cdot\delta(n) + \delta(n-1) - \frac{1}{3}\delta(n-2)$

$\Rightarrow X(z) = Z[x(n)]$

$= \frac{1}{2} Z[\delta(n)] + Z[\delta(n-1)] - \frac{1}{3}[\delta(n-2)]$

$= \frac{1}{2}\cdot 1 + z^{-1}\cdot Z[\delta(n)] - \frac{1}{3}\cdot z^{-2}\cdot Z[\delta(n)]$

$= \frac{1}{2} + z^{-1} - \frac{1}{3}\cdot z^{-2}$

ROC. Entire Z-Plane except $z = 0$.

Note :- $Z[\delta(n)] = 1$: unit impulse signal.

Que :- $x(n) = u(n-2)$

$\Rightarrow X(z) = Z[x(n)] = Z[u(n-2)]$

$= z^{-2}\cdot Z[u(n)] = z^{-2}\cdot\frac{1}{1-z^{-1}} = z^{-2}\cdot\frac{1}{1-\frac{1}{z}}$

$= z^{-2}\cdot\frac{1}{\frac{z-1}{z}} = z^{-2}\cdot\frac{z^{-1}}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$

Here ROC : $|z| > 1$

Que :- $x(n) = (n+0.5)\frac{1}{3}^{n}\cdot u(n)$

$\Rightarrow X(z) = Z[x(n)]$ ~~$= Z[(n+0.5)\frac{1}{3}^{n}\cdot u(n)]$~~

$= Z\left[n\cdot\left(\frac{1}{3}\right)^{n}\cdot u(n) + 0.5\left(\frac{1}{3}\right)^{n} u(n)\right]$

$$= Z\left[n.\left(\frac{1}{3}\right)^n u(n)\right] + 0.5\, Z\left[\left(\frac{1}{3}\right)^n . u(n)\right]$$

$$= z\left[n.\, x_1(n)\right] + 0.5 \times \frac{1}{1-\frac{1}{3}z^{-1}}$$

<u>NOTE</u> :- $Z\left[a^n . u(n)\right] = \frac{1}{1-a.z^{-1}} = \frac{z}{z-a}$

$$= -z\, \frac{dx_1(z)}{dz} + 0.5 \frac{z}{z-\frac{1}{3}} \rightarrow ①$$

$$\Rightarrow \frac{dx_1(z)}{dz} = \frac{d}{dz}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$\Rightarrow \frac{dx_1(z)}{dz} = \frac{\frac{dz}{dz}.\left(z-\frac{1}{3}\right) - z.\frac{d\left(z-\frac{1}{3}\right)}{dx}}{\left(z-\frac{1}{3}\right)^2}$$

$$= \frac{\left(z-\frac{1}{3}\right) - z.1}{\left(z-\frac{1}{3}\right)^2} = \frac{-1/3}{\left(z-\frac{1}{3}\right)^2}$$

$$\therefore x(z) = \frac{(-z)\left(-\frac{1}{3}\right)}{\left(z-\frac{1}{3}\right)^2} + \frac{0.5z}{z-\frac{1}{3}}$$

$$= \frac{z}{3\left(z-\frac{1}{3}\right)^2} + \frac{z}{2\left(z-\frac{1}{3}\right)}$$

<u>Que</u> (a) If $Y(z) = \frac{0.5\left(1-0.5z^{-1}\right)}{\left(1-0.25z^{-1}\right)\left(1-0.75z^{-1}\right)\left(1-z^{-1}\right)}$

Find the steady state value of $y(n)$ if it exists.

(b) Find $x(\infty)$ if $x(z)$ is given by $x(z) = \frac{3z}{(z-1)(z+1)}$

A) <u>Final Value Theorem</u> :-

According to final value Theorem,

$(z-1)\, x(z)$ or $(1-z^{-1})\, x(z)$ has all the Poles should lie inside the unit circle, then only final value of $x(n)$ will exist. That means no Pole should lie on the unit circle (or) outside the unit circle.

a) Steady state value of $y(n)$ is

$$y(\infty) = \lim_{z^{-1}\to 1} (1-z^{-1}) \cdot Y(z)$$

$$\lim_{z^{-1}\to 1} (1-z^{-1}) \cdot \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})(1-z^{-1})}$$

Here two Poles 0.25 & 0.75 lie inside the unit circle.

$$= \lim_{z^{-1}\to 1} \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})}$$

$$= \frac{0.5\times(1-0.5\times 1)}{(1-0.25\times 1)(1-0.75\times 1)} = \frac{0.5\times 0.5}{0.75\times 0.25}$$

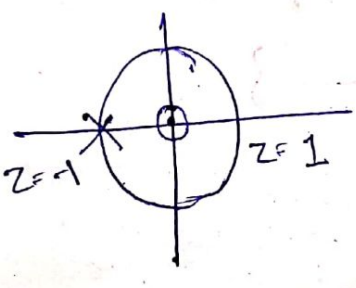
$$= \frac{0.25}{0.75\times 0.25} = \frac{1}{0.75} = 1.33$$

d) Steady state value of $x(n)$ is

$$x(\infty) = \lim_{z\to 1} (z-1) \cdot X(z)$$

$$= \lim_{z\to 1} (z-1) \frac{3z}{(z-1)(z+1)} = \lim_{z\to 1} \frac{3z}{z+1}$$

Here $(z-1)X(z) = \frac{3z}{z+1}$ has one Pole at $z=-1$ on the unit circle. So final value of $x(\infty)$ does not exist.



* The Inverse Z-Transform :-

→ It is expressed as $x(n) = Z^{-1}[X(z)]$.

By using 3 methods we can perform the inverse Z-transform.

① Long Division Method

② Partial Fraction Expansion Method

③ Residue method

① Long Division Method :-

Que Determine the inverse Z-transform of

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$ when

(i) ROC : $|z| > 1$; (ii) ROC : $|z| < \frac{1}{2}$.

A)

i) Here ROC is $|z| > 1$; that means outwards of the unit circle.

So, $x(n)$ is causal signal

$$\therefore X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \ldots$$

By taking Inverse Z-transform, we will get

$$x(n) = \{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \ldots\}$$

$n = 0$

$$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \;\Big|\; 1 \qquad \text{quotient: } 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \dots$$

$$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$$
$$(-)\quad(+)\qquad(-)$$
$$\frac{3}{2}z^{-1} - \frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3}$$
$$(-)\quad(+)\qquad(-)$$
$$\frac{7}{4}z^{-2} - \frac{3}{4}z^{-3}$$

$$\frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} + \frac{7}{8}z^{-4}$$
$$(-)\quad(+)\qquad(-)$$
$$\frac{15}{8}z^{-3} - \frac{7}{8}z^{-4}$$

ii) In this case ROC is $|z| < 0.5$; that means the interior of the circle. Here $x(n)$ signal is anti-causal signal. Here we will get a power series expansion is +ve powers of z. We perform the long division method in following way:-

$$\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \;\Big|\; 1 \qquad \text{quotient: } 2z^{2} + 6z^{3} + 14z^{4} + \dots$$

$$1 - 3z + 2z^{2}$$
$$(-)\quad(+)\qquad(-)$$
$$3z - 2z^{2}$$

$$3z - 9z^{2} + 6z^{3}$$
$$(-)\quad(+)\qquad(-)$$
$$7z^{2} - 6z^{3}$$

$$7z^{2} - 21z^{3} + 14z^{4}$$
$$(-)\quad(+)\qquad(-)$$
$$15z^{3} - 14z^{4}$$

$$\therefore x(z) = \frac{1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1}$$

$$= 0 + 0 \cdot z^{-1} + 2z^{2} + 6z^{3} + 14z^{4} + \dots$$

Take inverse Z-transform is

$x(n) = \{\ldots\ldots 14, 6, 2, 0, 0\}$
$\uparrow$ $n=0$

Anti-causal signal (or) left side signal

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Que :- The impulse response for a discrete time system is given as $h(n) = \{1, 2, 3\}$ & o/p response is given as $y(n) = \{1, 1, 2, -1, 3\}$. Determine discrete time i/p signal.

A)
Method 1

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$y(n) = h(n) * x(n)$

Let $x(n) = \{a, b, c\}$.

$x(n) \backslash h(n)$	1	2	3
a	a	2a	3a
b	b	2b	3b
c	c	2c	3c

$y(n) = \{a, b+2a, 3a+2b+c, 2c+3b, 3c\}$

$= \{1, 1, 2, -1, 3\}$

By comparing, $a = 1$; $b + 2a$

$\Rightarrow b + 2.1 = 1 \Rightarrow b = -1.$; $c = 1$

$\therefore x(n) = \{a, b, c\} = \{1, -1, 1\}$.

Method 2

$y(n) = x(n) * h(n)$

Make Z transform on both sides.

$\Rightarrow Y(z) = X(z) . H(z)$.

$\Rightarrow X(z) = \dfrac{Y(z)}{H(z)}$

$\therefore H(z) = \sum_{n=0}^{2} h(n).z^{-n}$

$= 1 + 2.z^{-1} + 3.z^{-2}$.

$\therefore Y(z) = \sum_{n=0}^{4} y(n).z^{-n} = 1 + z^{-1} + 2.z^{-2} - z^{-3} + 3.z^{-4}$.

$$
\begin{array}{r|l}
 & 1 - z^{-1} + z^{-2} \\
\hline
1+2z^{-1}+3z^{-2} & 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} \\
 & 1 + 2z^{-1} + 3z^{-2} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & -z^{-1} - z^{-2} - z^{-3} + 3z^{-4} \\
 & -z^{-1} - 2z^{-2} - 3z^{-3} \\
 & (+)\quad(+)\quad(+) \\
\hline
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & 0
\end{array}
$$

$\therefore X(z) = 1 - z^{-1} + z^{-2}$

By making inverse z-transform

$x(n) = \{1, -1, 1\}$

② <u>Inverse z-transform by using Partial Fraction Method</u> :-

→ ~~If H(z) can~~ Here factorization is done in denominator.

→ If $H(z)$ can be written as $H(z) = \dfrac{A_1}{z-p} + \dfrac{A_2}{(z-p)^2} + \dots \dfrac{A_{m-1}}{(z-p)^{m-1}} + \dfrac{A_m}{(z-p)^m}$

Then $A_m = \lim\limits_{z \to p} (z-p)^m \cdot H(z)$

$$A_{m-1} = \frac{1}{1!} \lim_{z \to p} \frac{d^1}{dz^1}\left[(z-p)^m \cdot H(z)\right]$$

$$A_{m-2} = \frac{1}{2!} \lim_{z \to p} \frac{d^2}{dz^2}\left[(z-p)^m \cdot H(z)\right]$$

$$A_{m-3} = \frac{1}{3!} \lim_{z \to p} \frac{d^3}{dz^3}\left[(z-p)^m \cdot H(z)\right]$$

$$A_3 = \frac{1}{(m-3)!} \lim_{z \to p} \cdot \frac{d^{m-3}}{dz^{m-3}} \left[(z-p)^{m} \cdot H(z) \right]$$

$$A_2 = \frac{1}{(m-2)!} \lim_{z \to p} \frac{d^{m-2}}{dz^{m-2}} \left[(z-p)^m \cdot H(z) \right]$$

$$A_1 = \frac{1}{(m-1)!} \lim_{z \to p} \frac{d^{m-1}}{dz^{m-1}} \left[(z-p)^m \cdot H(z) \right]$$

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Que:- By using partial fraction method. find Inverse Z. transform of the following transfer function.

$$H(z) = \frac{-4+8 \cdot z^{-1}}{1+6z^{-1}+8z^{-2}}.$$

A) Given transfer function is $H(z) = \frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$

$$= \frac{-4+8z^{-1}}{1+4z^{-1}+2z^{-1}+8z^{-2}} = \frac{-4+8z^{-1}}{(1+4z^{-1})+2z^{-1}(1+4z^{-1})}$$

$$= \frac{-4+8z^{-1}}{(1+2z^{-1})(1+4z^{-1})}$$ (∵ By making Partial fraction)

$$\Rightarrow H(z) = \frac{A}{1+2z^{-1}} + \frac{B}{1+4z^{-1}}$$

$$A = \lim_{z^{-1} \to -1/2} (1+2z^{-1}) \cdot H(z)$$

$$= \lim_{z^{-1} \to -1/2} \cancel{(1+2z^{-1})} \cdot \frac{(-4+8z^{-1})}{\cancel{(1+2z^{-1})}(1+4z^{-1})}$$

$$= \frac{-4+8\times(-1/2)}{1+4\times(-1/2)} = \frac{-8}{-1} = 8.$$

$$B = \lim_{z^{-1}\to -1/4} (1+4z^{-1}).\,H(z)$$

$$= \lim_{z^{-1}\to -1/4} \cancel{(1+4z^{-1})}.\,\frac{(-4+8z^{-1})}{(1+2z^{-1})\cancel{(1+4z^{-1})}}$$

$$= \frac{-4+8\times(-1/4)}{1+2\times(-1/4)} = \frac{-6}{1/2} = -12.$$

$$\therefore H(z) = \frac{8}{1+2z^{-1}} + \frac{-12}{1+4z^{-1}}$$

By making Inverse Z-transform,

$$h(n) = 8.(-2)^n.u(n) - 12\,(-4)^n\,u(n)$$

Que By using Partial fraction method find inverse Z-transform: $x(z) = \dfrac{z^3}{(z+1)(z-1)^2}$

1) Given that, $\dfrac{X(z)}{z} = \dfrac{z^2}{(z+1)(z-1)^2}$.

Take $F(z) = \dfrac{X(z)}{z} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

$$A = \lim_{z\to -1} (z+1).\,F(z)$$

$$= \lim_{z\to -1} \cancel{(z+1)}.\,\frac{z^2}{\cancel{(z+1)}(z-1)^2} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$\boxed{\begin{aligned} Z[a^n.u(n)] &= \frac{1}{1-az^{-1}} = \frac{z}{z-a} \\ Z[(-a)^n.u(n)] &= \frac{1}{1-(-a)z^{-1}} = \frac{1}{1+az^{-1}} \\ &= \frac{z}{z+a} \end{aligned}}$$

$$C = \lim_{z\to 1} (z-1)^2.\,F(z)$$

$$= \lim_{z\to 1} \cancel{(z-1)^2}.\,\frac{z^2}{(z+1)\cancel{(z-1)^2}} = \frac{1^2}{1+1} = \frac{1}{2}$$

$$B = \frac{1}{1!}\lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2.\,F(z)\right]$$

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$$= \lim_{z \to 1} \frac{d}{dz}\left[\frac{(z-1)^2 \cdot z^2}{(z+1)(z-1)^2}\right]$$

$$\Rightarrow B = \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^2}{z+1}\right] \qquad \left[\because \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}\right]$$

$$= \lim_{z \to 1} \frac{\frac{dz^2}{dz}(z+1) - z^2 \cdot \frac{d(z+1)}{dz}}{(z+1)^2}$$

$$= \lim_{z \to 1} \frac{2 \cdot z(z+1) - z^2 \cdot 1}{(z+1)^2} = \lim_{z \to 1} \frac{2z^2 + 2z - z^2}{(z+1)^2}$$

$$= \lim_{z \to 1} \frac{z^2 + 2z}{(z+1)^2} = \frac{1^2 + 2 \times 1}{(1+1)^2} = 3/4$$

$$\therefore F(z) = \frac{x(z)}{z} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\Rightarrow \frac{x(z)}{z} = \frac{1}{4} \cdot \frac{1}{z+1} + \frac{3}{4} \cdot \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{(z-1)^2}$$

$$\Rightarrow x(z) = \frac{1}{4} \cdot \frac{z}{z+1} + \frac{3}{4} \cdot \frac{z}{z-1} + \frac{1}{2} \cdot \frac{z}{(z-1)^2}$$

Make inverse z-transform on both sides, we get,

$$\rightarrow x(n) = \frac{1}{4} \cdot (-1)^n u(n) + \frac{3}{4} \cdot (1)^n \cdot u(n) + \frac{1}{2} \cdot n(1)^n \cdot u(n)$$

$$\Rightarrow x(n) = \frac{1}{4}(-1)^n \cdot u(n) + \frac{3}{4} \cdot u(n) + \frac{1}{2} n \cdot u(n)$$

3.7.3 Residue Method

In this method, we obtain, inverse z-transform x [n], by summing residues of [X (z) z^{n-1}] at all poles. Mathematically, this may be expressed as

$$x(n) = \sum_{\substack{\text{all poles} \\ X(z)}} \text{residues of } (X(z)\, z^{n-1}] \quad \ldots(3.46)$$

Here, the residue for any pole of order m at z = β is

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to \beta} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-\beta)^m X(z).z^{n-1} \right] \right\} \quad \ldots(3.47)$$

Example 3.35

Use residue method to find the inverse z-transform, x (n) for

$$X(z) = \frac{z}{(z-1)(z-2)}$$

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Solution: The given transform is

$$X(z) = \frac{z}{(z-1)(z-2)}$$

X(z) has two poles of order m = I at z = 1 and at z = 2.

We can obtain the corresponding residues as ahead :

For poles at z = 1

$$\text{Residue} = \frac{1}{0!}\lim_{z\to 1}\left\{\frac{d^0}{dz^0}\left[\cancel{(z-1)^1}.\frac{z.z^{n-1}}{\cancel{(z-1)}(z-2)}\right]\right\}$$

(1-1)

$$\text{or} \qquad \text{Residue} = \lim_{z\to 1}\left[\frac{z}{z-2}.z^{n-1}\right] = \lim_{z\to 1}\left[\frac{(z^1).z^n.z^{-1}}{z-2}\right] = \lim_{z\to 1}\left[\frac{z^n}{z-2}\right]$$

$$\text{or} \qquad \text{Residue} = -1 \qquad = \frac{1}{1-2} = 1/-1$$

Similarly, for poles at z = 2

$$\text{Residue} = \frac{1}{0!}\lim_{z\to 2}\left\{\frac{d^0}{dz^0}\left[\cancel{(z-2)^1}\frac{z.z^{n-1}}{(z-1)\cancel{(z-2)}}\right]\right\}$$

(4-1)

$$= \lim_{z\to 2}\left[\frac{z}{z-1}z^{n-1}\right] = 2.2^{n-1} = \left[\frac{z^1 . z^n . z^{-1}}{z-1}\right] = \frac{2^n}{2-1} = 2^n.$$

Residue = 2^n

Hence, $x(n) = \{-1 + 2^n\}.\ u[n]$

Example 3.37

Obtain the inverse z-transform of

$$X(n) = ln\ (1 + az^{-1}), |z| > |a|$$

Solution : **According to logarithmic series expansion, we have**

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots.$$

Therefore, $X(z) = ln(1 + az^{-1})$

Simplifying, we have

$$X(z) = az^{-1} - \frac{1}{2}(az^{-1})^2 + \frac{1}{3}(az^{-1})^3 \ldots$$

$$= az^{-1} - \frac{1}{2}a^2z^{-2} + \frac{1}{3}a^3z^{-3} \ldots$$

Taking inverse z-transform, we obtain

$$x(n) = \left\{ \underset{\uparrow}{0}, a, -\frac{1}{2}a^2, \frac{1}{3}a^3, \right\}$$ Ans.

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Example 3.39

Find the inverse z-transform of

$$X(z) = \frac{z^3 + z^2}{(z-1)(z-3)} \qquad \text{ROC} : |z| > 3$$

.+ z =

Solution

$$\frac{X(z)}{z} = \frac{z^2+z}{(z-1)(z-3)}$$; Here 2 poles are present, having order m = 1

Converting the above improper rational function ($\because$ M = N) into sum of a constant and a proper rational function we get

$$\frac{X(z)}{z} = 1 + \frac{5z-3}{(z-1)(z-3)}$$

The rational expression can be expanded by Partial fraction expansion

$$\frac{5z-3}{(z-1)(z-3)} = \frac{C_1}{z-1} + \frac{C_2}{z-3}$$

where

$$C_1 = (z-1)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=1} = -1$$

$$C_2 = (z-3)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=3} = 6$$

Therefore

$$\frac{X(z)}{z} = 1 - \frac{1}{z-1} + \frac{6}{z-3}$$

$$X(z) = z - \frac{z}{z-1} + \frac{6z}{z-3}$$

Taking Inverse z-transform on both sides we get

$$x(n) = \delta(n+1) - u(n) + 6(3)^n u(n)$$

Example 3.40

Use the residue method to find the inverse z-transform of

$$X(z) = \frac{z}{(z-2)(z-3)} \; |z| < 2$$

Solution :

In this case there are two poles z = 3 and z = 2 outside the ROC | z | < 2, so the sequence is non causal.

For n < 0

Example 3.41

Find the inverse z-transform of $X(z) = \dfrac{z^2 + z}{(z-1)(z-3)}$, ROC : $|z| > 3$. Using (a) Partial fraction expansion method (b) Residue method (c) Convolution Method.

Solution

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Ex:- 3.41

(b)

A) Here ROC is $|z| > 3$, so the sequence is causal signal. Because ROC is outward of the outermost pole.

Here two Poles available $z = 1, 3$ having order $m = 1$.

Residue at Pole $z = 1$ having order $m = 1$:-

$$R_1 = \lim_{z \to 1} \left[(z-1) \cdot X(z) \cdot z^{n-1} \right]$$

$$= \lim_{z \to 1} \left[(z-1) \cdot \frac{z^2+z}{(z-1)(z-3)} \cdot z^{n-1} \right] = \lim_{z \to 1} \left[\frac{z^1(z+1) \cdot z^n \cdot z^{-1}}{z-3} \right]$$

$$= \lim_{z \to 1} \left[\frac{(1+1) \cdot (1)^n}{1-3} \right] = \frac{2 \cdot 1}{-2} = -1.$$

Residue at Pole $z=3$;

$$R_2 = \lim_{z \to 3} \left[(z-3) \cdot x(z) \cdot z^{n-1} \right]$$

$$= \lim_{z \to 3} \left[(z-3) \cdot \frac{z^2+z}{(z-1)(z-3)} \cdot z^{n-1} \right]$$

$$= \lim_{z \to 3} \left[\frac{z^1(z+1) \cdot z^n \cdot z^{-1}}{(z-1)} \right] = \frac{3+1 \cdot 3^n}{3-1} = \frac{4(3)^n}{2}$$

$$\therefore x(n) = R_1 + R_2 = \left[-1 + 2(3)^n \right] u(n)$$

Example 3.45

Find the system function, H(z) and unit-sample response h(n) of the system whose difference equation is described as under :

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

where y(n) and x(n) are the output and input of the system, respectively.

Solution : The given difference equation is

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

Taking the z-transform of above difference equation, we get

$$y(z) = \frac{1}{2}z^{-1}Y(z) + 2X(z)$$

or $$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = 2X(z)$$

Here $x(n)$ = unit impulse $\delta(n)$

or $$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}} =$$

$x(z) = Z[\delta(n)] = 1$

So o/p is impulse response $y(n) = h(n)$.

This system function has a pole at $z = \frac{1}{2}$ and zero at $z = 0$.

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$ = system function

Also,

$$h(n) = \text{Inverse z-transform of } H(z) = Z^{-1}\left[\frac{2}{1 - \frac{1}{2}z^{-1}}\right]$$

or, $$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

This is the unit-sample response of the system. Ans.

Example 3.49

Determine the causal signal x(n) having the z-transform

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

P-173 Ex :- 3.49.

A) By using Residue Method

$$x(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2} = \frac{1}{\left(1-\frac{2}{z}\right)\left[1-\frac{1}{z}\right]^2}$$

$$= \frac{1}{\left(\frac{z-2}{z}\right)\left(\frac{z-1}{z}\right)^2} = \frac{1}{\left(\frac{z-2}{z}\right)\cdot\left[\frac{(z-1)^2}{z^2}\right]} = \frac{1}{\frac{(z-2)(z-1)^2}{z^3}}$$

$$= \frac{z^3}{(z-2)(z-1)^2}$$; Here $x(z)$ is having two poles at $z=2$ having order 1, at $z=1$, having order $m=2$.

Residue at pole $z=2$:-

$$R_1 = \lim_{z\to 2}\left[(z-2).\,x(z).\,z^{n-1}\right] = \lim_{z\to 2}\left[z-2.\,\frac{z^3}{(z-2)(z-1)^2}.\,z^{n-1}\right]$$

$$= \lim_{z\to 2}\left[\frac{z^3.\,z^n * z^{-1}}{(z-1)^2}\right] = \lim_{z\to 2}\left[\frac{z^2.\,z^n}{(z-1)^2}\right]$$

$$= \frac{2^2.\,2^n}{(2-1)^2} = 4.\,2^n$$

Residue at pole $Z = 1$ having order $m = 2$:

$0! = 1$
$1! = 1$

$R_2 = \cancel{\lim_{Z \to 1}} [Z$

$$R_2 = \frac{1}{(m-1)!} \lim_{Z \to 1} \frac{d^{m-1}}{dz^{m-1}} \left[(Z-1)^m \cdot X(Z) \cdot Z^{n-1} \right]$$

$\frac{d x^n}{dx} = n \cdot x^{n-1}$

Here $m = 2$

$$= 1 \cdot \lim_{Z \to 1} \frac{d}{dz} \left[(Z-1)^2 \cdot \frac{Z^3}{(Z-2)(Z-1)^2} \cdot Z^{n-1} \right]$$

$$= \lim_{Z \to 1} \frac{d}{dz} \left[\frac{Z^3 \cdot Z^{-1} \cdot Z^n}{Z-2} \right] = \lim_{Z \to 1} \frac{d}{dz} \left[\frac{Z^{n+2}}{Z-2} \right]$$

$$= \lim_{Z \to 1} \frac{\frac{d Z^{n+2}}{dz} \cdot (Z-2) - Z^{n+2} \cdot \frac{d(Z-2)}{dz}}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{(n+2) \cdot Z^{n+2-1} \cdot (Z-2) - Z^{n+2} \cdot 1}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{n+2 \cdot Z^{n+1} \cdot (Z-2) - Z^{n+2}}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{n+2 \cdot Z^n \cdot Z^1 \cdot (Z-2) - Z^n \cdot Z^2}{(Z-2)^2}$$

$$= \frac{n+2 \cdot 1^n \cdot 1^1 \cdot (1-2) - 1^n \cdot 1^2}{(1-2)^2} = \frac{-(n+2) - 1}{1}$$

$$= -n - 3 = -(n+3)$$

$\therefore \; x(n) = R_1 + R_2$

$$= \left[4(2)^n - (n+3) \right] u(n)$$

NOTE:- $Z[x(n-1)] = Z^{-1} X(Z) + x(-1)$

where $x(-1)$ represent the inital condition.

$$Z[x(n-2)] = Z^{-2} X(Z) + Z^{-1} x(-1) + x(-2)$$

$$z[x(n-3)] = z^{-3} \cdot X(z) + z^{-2} \cdot x(-1) + z^{-1} \cdot x(-2) + x(-3)$$

where $x(-1)$, $x(-2)$, $x(-3)$ are initial conditions.

Que. Solve the difference equation

$y(n) + 3y(n-1) = x(n)$, where $x(n)$ is $\left(\frac{1}{2}\right)^n \cdot u(n)$

& $y(-1) = 2$.

A) Apply Z-transform on both sides to the given difference equation:

$$Y(z) + 3\left[z^{-1} \cdot Y(z) + y(-1)\right] = X(z)$$

$$\Rightarrow Y(z) + 3\left[z^{-1} Y(z) + 2\right] = X(z)$$

$$\Rightarrow Y(z) + 3z^{-1} \cdot Y(z) + 6 = X(z)$$

$$\Rightarrow Y(z)\left[1 + 3 \cdot z^{-1}\right] = -6 + X(z)$$

$$\Rightarrow Y(z) = \frac{-6}{1+3z^{-1}} + \frac{X(z)}{(1+3z^{-1})} = \frac{-6}{1+3z^{-1}} + \frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$\Rightarrow Y(z) = \frac{-6}{1+3z^{-1}} + \frac{A}{1+3z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \lim_{z^{-1} \to -1/3} (1+3z^{-1}) \frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{1}{1-\frac{1}{2} \times \left(-\frac{1}{3}\right)} = \frac{1}{1+1/6} = \frac{6}{7}$$

$$B = \lim_{z^{-1} \to 2} \left(1-\frac{1}{2}z^{-1}\right) \cdot \frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{1}{1+3 \times 2} = \frac{1}{1+6} = 1/7$$

$$\therefore Y(z) = -6 \cdot \frac{1}{1+3z^{-1}} + \frac{6}{7} \cdot \frac{1}{1+3z^{-1}} + \frac{1}{7} \cdot \frac{1}{1-\frac{1}{2}z^{-1}}$$

Applying z-inverse transform on both sides we will get

$$\Rightarrow y(n) = -6\,(-3)^n \cdot u(n) + \frac{6}{7}\,(-3)^n \cdot u(n) + \frac{1}{7}\left(\frac{1}{2}\right)^n \cdot u(n)$$

$$\Rightarrow y(n) = \frac{-36}{7} \cdot (-3)^n \cdot u(n) + \frac{1}{7} \cdot \left(\frac{1}{2}\right)^n \cdot u(n)$$

$X(z)$

Example 3.54

Determine the impulse response and the step response of the following causal system. Determine it is stable or not.

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

Solution

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

Inpulse response means $x(n) = \sigma(n)$, here $y(n) = h(n)$

Applying Z - transform to equation (1),

$$Y(z) = \frac{3}{4}Z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z)$$

$$\Rightarrow Y(z)\left[1 - \frac{3}{4}z + \frac{2}{8}z^{-2}\right] = X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z} - 2$$

$$\Rightarrow H(z) = \frac{8z^2}{8z^2 - 6z + 1}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{8z}{8z^2 - 6z + 1}$$

$$= \frac{8z}{(2z-1)(4z-1)}$$

$$= \frac{A}{1z-1} + \frac{B}{4z-1}$$

$$A = \left.\frac{8z}{4z-1}\right|_{z=\frac{1}{2}} = 4$$

$$B = \left.\frac{8z}{2z-1}\right|_{z=\frac{1}{4}} = -4$$

$$\frac{H(z)}{z} = \frac{4}{2z-1} - \frac{4}{4z-1} \Rightarrow 1 + (z) = \frac{4z}{2z-1} - \frac{4z}{4z-1}$$

$$\Rightarrow h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u(n)$$

For step response, $x(n) = u(n)$

$$\Rightarrow X(z) = \frac{Z}{z-1}$$

So Y(z) = H(z) X(z)

$$=\left[\frac{4z}{2z-1}-\frac{4z}{4z-1}\right]\cdot\frac{z}{z-1}$$

$$=\frac{8z^2}{(z-1)(2z-1)(4z-1)}$$

$$\Rightarrow\frac{Y(z)}{z}=\frac{8z}{(z-1)(2z-1)(4z-1)}$$

$$=\frac{A}{z-1}+\frac{B}{2z-1}+\frac{C}{4z-1}$$

$$A=\left.\frac{8z}{(2z-1)(4z-1)}\right|_{Z=1}=\frac{8}{3}$$

$$B=\left.\frac{8z}{(z-1)(4z-1)}\right|_{Z=\frac{1}{2}}$$

$$=-8$$

$$=\frac{2}{\left(\frac{-3}{4}\right)\left(\frac{-1}{2}\right)}$$

$$=\frac{16}{3}$$

$$\text{So } Y(z)=\frac{8}{3}\frac{z}{z-1}-4\frac{2z}{2z-1}+\frac{4}{3}\frac{4z}{4z-1}$$

$$y(n)=\frac{8}{3}u(n)-4\left(\frac{1}{2}\right)^n u(n)+\frac{4}{3}\left(\frac{1}{4}\right)^n \text{un}$$

Example 3.55

We want to design a causal discrete-time LTI system with the property that if the input is.

$$x(n)=\left(\frac{1}{2}\right)^n u(n)-\frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

then the output is

$$y(n)=\left(\frac{1}{3}\right)^n u(n)$$

(a) Determine the impulse response h(n) and the system function H(z) of a system that satisfies the foregoing conditions.

(b) Find the difference equation that characterizes this system.

(c) Determine if the system is stable.

Solution :

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$y(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4} z^{-1} \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{2z}{2z-1} - \frac{1}{2}\frac{1}{2z-1}$$

$$= \frac{4z-1}{2(2z-1)}$$

$$y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{1}{1-\frac{1}{3z}}$$

$$Y(z) = \frac{3z}{3z-1}$$

(a) $$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{6z(2z-1)}{(3z-1)(4z-1)}$$

$$\Rightarrow \frac{H(z)}{Z} = \frac{6(2z-1)}{(3z-1)(4z-1)}$$

$$= \frac{A}{3z-1} + \frac{B}{4z-1}$$

$$A = \left.\frac{6(2z-1)}{4z-1}\right|_{z=\frac{1}{3}}$$

$$= \frac{6.\left(\frac{-1}{3}\right)}{\frac{4}{3}-1} = -6$$

$$B = \left.\frac{6(2z-1)}{3z-1}\right|_{z=\frac{1}{4}}$$

$$= \frac{6\left(\frac{-1}{2}\right)}{\frac{-1}{4\,2}} = 12$$

$$\boxed{H(z)\frac{-6z}{3z-1}+\frac{12z}{4z-1}}$$ Ans (a)

$$\Rightarrow \boxed{h(n) = -2.\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{4}\right)^n u(n)}$$ Ans (a)

(b) $$H(z) = \frac{Y(z)}{X(z)} = \frac{6(z)(2z-1)}{(3z-1)(4z-1)}$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{12z^2-6z}{12z^2-7z+1}$$

$$\Rightarrow 12z^2Y(z) - 7zY(z) + Y(z) = 12z^2X(z) - 6zX(z)$$

So difference equation is,

$12y(n + 2) - 7\,y(n + 1) + y(n) = 12x\,(n + 2) - 6x\,(n + 1)$

(c) $$\sum_{n=-\infty}^{\infty} |h(n)|$$

$$= \left| -2\frac{1}{1-\frac{1}{3}} + 3\frac{1}{1-\frac{1}{4}} \right|$$

$$= |-3+4| = 1 \leq \infty$$

so it is a stable system.