Notes on Digital Signal Processing

by

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SAMPLING THEORY :- ①

① It is a process to convert continuous time signals in to Discrete signal.

② Sufficient number of samples must be taken, so that the original signal is reconstructed properly.

③ Number of samples to be taken depends on Maximum signal Frequency present in the signal.

④ Different Types of sampling are:
   ⓐ Ideal samples, ⓑ Natural samples, ⓒ Flat Top samples.

Statement of Sampling Theorem :-

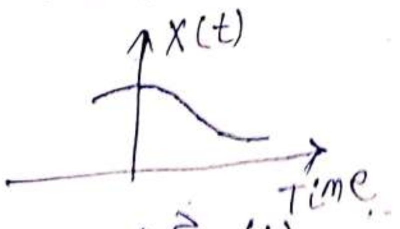
(i) A band limited signal of Finite energy, which has no Frequency component higher than f_m (HZ), is completely described by its sample values at uniform intervals less than (or) equal to $\frac{1}{2f_m}$.

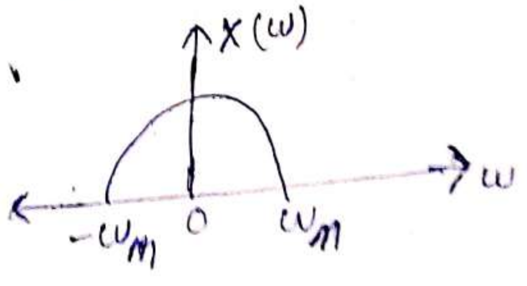
$$\boxed{T_s \leq \frac{1}{2f_m}}$$

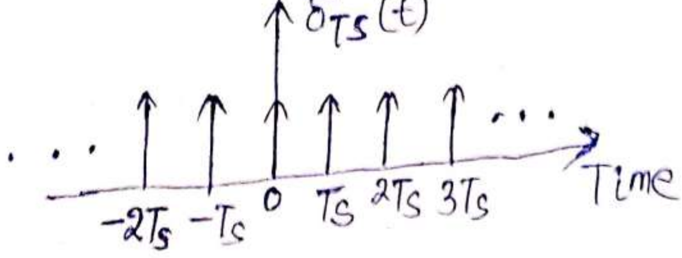
(ii) A Band limited signal of Finite energy, which has no Frequency components higher than f_m (HZ), may be completely recovered from the knowledge of its samples taken at the rate of $2f_m$ samples per second.

$$\boxed{F_s \geq 2f_m}$$

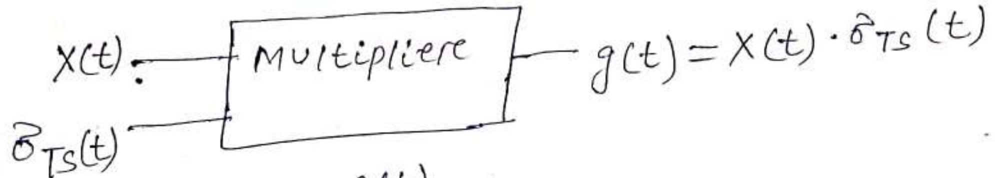
↳ If the signal is band limited to f_m then
$X(\omega) = 0$, for $\omega > \omega_m$

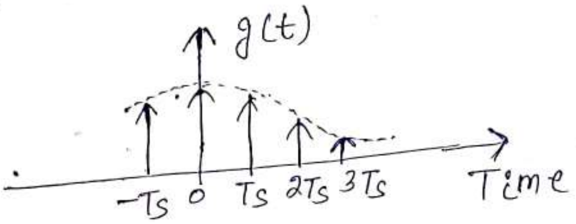




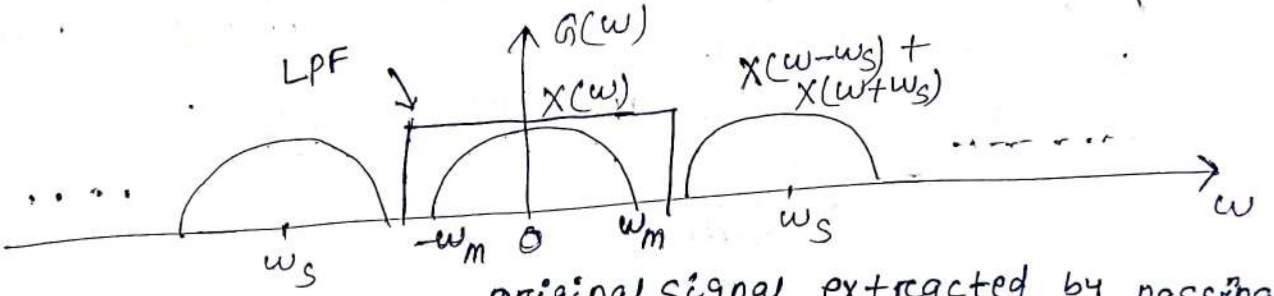


$$\delta_{Ts}(t) = \frac{1}{T_s}\left[1 + 2\cdot\cos\omega_s t + 2\cdot\cos 2\omega_s t + 2\cdot\cos 3\omega_s t + \cdots\right]$$





↳ In Frequency Domain, $G(\omega) = \frac{1}{T_s}\left[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s) + \cdots\right]$



original signal extracted by passing through LPF.

$\boxed{\omega_s > 2\omega_m}$

↳ As long as $f_s > 2f_m$, $G(\omega)$ will repeat periodically without overlapping.

↳ Spectrum $G(\omega)$ extends upto ∞ (infinite) frequency but our purpose is to extract original spectrum $X(\omega)$ out of the spectrum $G(\omega)$.

↳ At receiver we place LPF of frequency ω_m. So we can extract original information.

↳ $f_s > 2f_m$, To avoid successive cycles not to overlape.
$f_s = 2f_m$, successive cycles just touch each other.
$f_s < 2f_m$, successive cycles overlape each other.

↳ Hence, for reconstruction without distortion

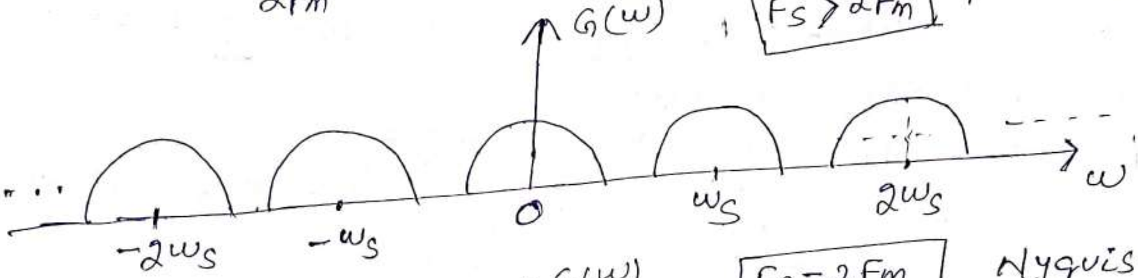
$\boxed{f_s \geq 2f_m}$

②

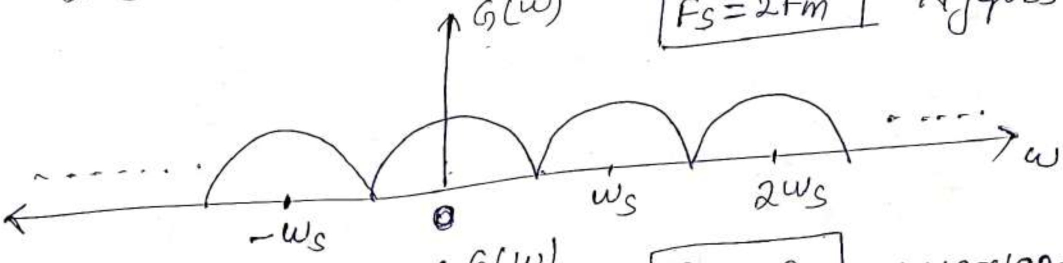
↳ $F_s = 2f_m$, Hence F_s is reefereeed as Nyquist Rate.

$T_s = \frac{1}{2f_m}$, T_s is Nyquist Interval

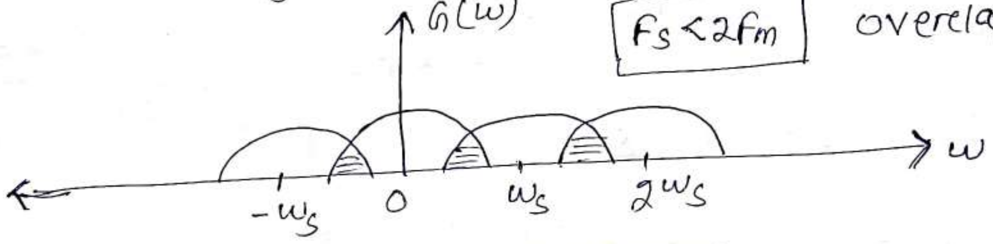
$\boxed{F_s > 2f_m}$ No overelapping



$\boxed{F_s = 2F_m}$ Nyquist sampling



$\boxed{F_s < 2f_m}$ overelape



↳ If $F_s < 2f_m$, then successive samples cycles of G(ω) will overelape each othere.

↳ Due to Aliasing effect, it is not possible to recover original signal x(t) by LPF.

↳ Hence due to overelape of one region to other region, signal x(t) is distorted.

↳ So, before we go for sampling, we pass original signal through LPF. This is even reeferreed as pre-alias Filter, other name is Band limit Filter.

↳ In short, to avoid aliasing:

① pre aliasing Filter can be used.

② $F_s \geq 2f_m$

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Examples based on sampling and Nyquist Rate :-

Que:- $x(t) = 3\cdot\cos(50\pi t) + 10\cdot\sin(300\pi t) - \cos(100\pi t)$
Calculate the Nyquist Rate for this signal.

Sol^n:- $f_1 = \frac{\omega_1}{2\pi} = 25\,Hz$, $f_2 = \frac{\omega_2}{2\pi} = 150\,Hz$, $f_3 = \frac{\omega_3}{2\pi} = 50\,Hz$

Maximum Frequency $f_m = 150\,Hz$

Nyquist Rate $F_s = 2f_m = 2\times 150 = 300\,Hz$

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Que:- Find the Nyquist Rate and Nyquist Interval for the signal $x(t) = \frac{1}{2\pi}\cdot\cos(4000\pi t)\cdot\cos(1000\pi t)$

Sol^n:- $x(t) = \frac{1}{4\pi}\left[2\cdot\cos(4000\pi t)\cdot\cos(1000\pi t)\right]$

$= \frac{1}{4\pi}\left[\cos(3000\pi t) + \cos(5000\pi t)\right]$

$f_1 = \frac{\omega_1}{2\pi} = 1500\,Hz$, $f_2 = \frac{\omega_2}{2\pi} = 2500\,Hz$, Maximum Frequency $f_m = 2500\,Hz$, Nyquist Rate $F_s = 2f_m = 2\times 2500 = 5000\,Hz$,

Nyquist Interval $T_s = \frac{1}{F_s} = \frac{1}{5000} = \boxed{0.2\,msec}$

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Que:- Deteremine the Nyquist reate for a continuous time signal $x(t) = 6\cdot\cos 50\pi t + 20\cdot\sin 300\pi t - 10\cdot\cos 100\pi t$.

Sol^n:- $f_1 = \frac{\omega_1}{2\pi} = 25\,Hz$, $f_2 = \frac{\omega_2}{2\pi} = 150\,Hz$, $f_3 = \frac{\omega_3}{2\pi} = 50\,Hz$

Maximum Frequency $f_m = 150\,Hz$

Nyquist Rate $F_s = 2f_m = 300\,Hz$

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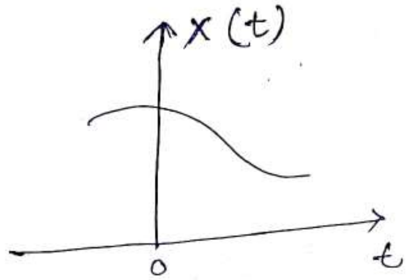
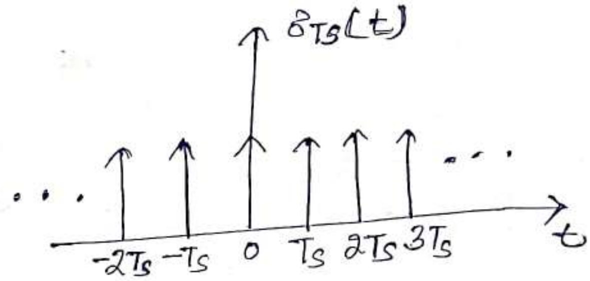
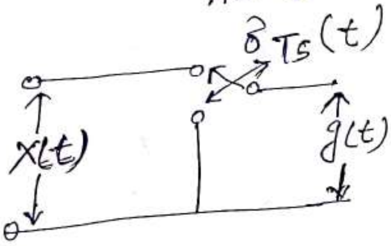
Instantaneous sampling (or) Impulse sampling (or) Ideal sampling :-

↳ It uses principle of multiplication.

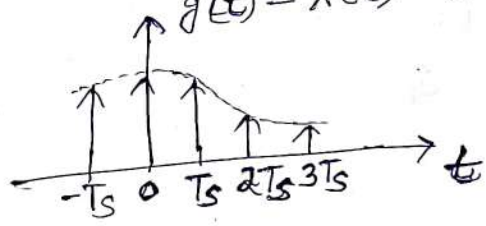




$$\delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$g(t) = x(t) \cdot \delta_{Ts}(t)$



↳ To generate ideal samples train, we use switching sampler. ↳ If we assume, closing time $t \to 0$, then it has to be consider ideal impulse train.

↳ Impulse Train $\delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

↳ output $g(t) = x(t) \cdot \delta_{Ts}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

↳ In Frequency Domain, $G(\omega) = F_s \cdot \sum_{n=-\infty}^{\infty} X(F - nF_s)$

↳ practically, This method is not possible. High noise interference is available. signal Energy is very Low.

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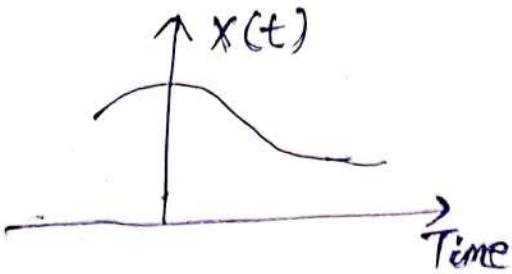
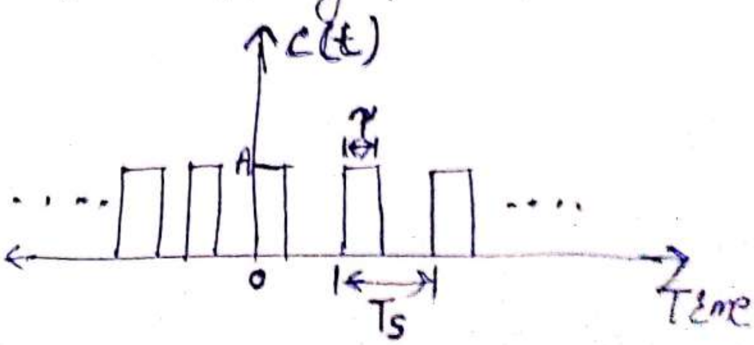
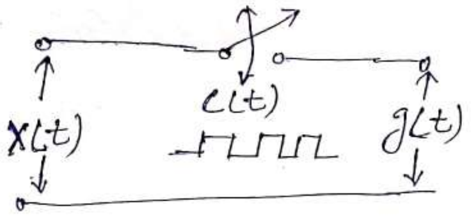
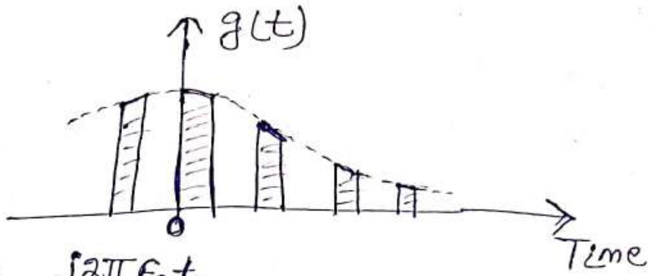
NATURAL SAMPLING :- ① It uses chopping principle.









$$c(t) = \frac{\tau \cdot A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}(f_n \tau) \cdot e^{j2\pi f_s t}$$

↳ $g(t) = x(t)$, for $c(t) = A$

$g(t) = 0$, for $c(t) = 0$

↳ So, Mathematically $g(t) = x(t) \cdot c(t)$

$$= \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} x(t) \cdot \text{sinc}(f_n \cdot \tau) \cdot e^{j2\pi f_s t}$$

↳ Frequency Domain

$$G(\omega) = \frac{\tau \cdot A}{T_s} \sum_{n=-\infty}^{\infty} \sin(nf_s \tau) \cdot X(f - nf_s)$$

↳ This method is used practically. Noise interference is less. Because g(t) sampled output impulses have finite pulse duration (τ) and Finite Energy.

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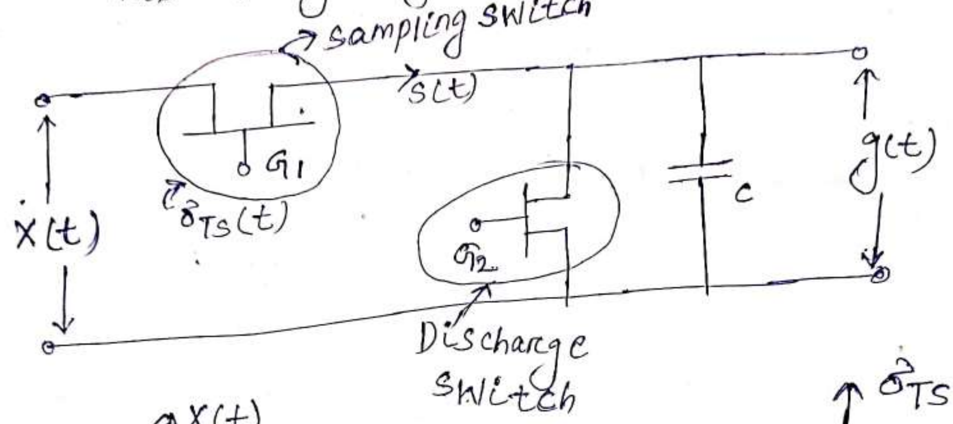
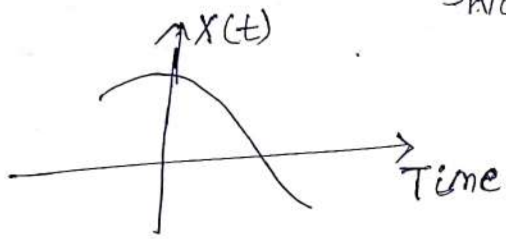
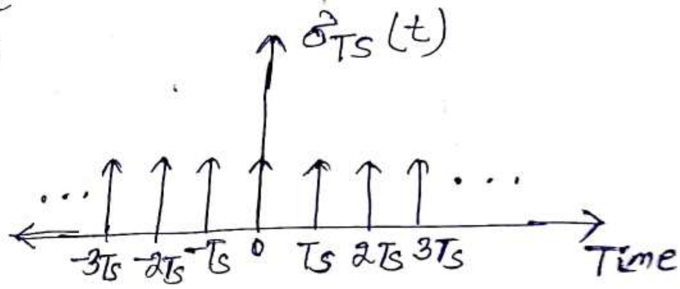
FLAT-TOP SAMPLING (PAM) :-

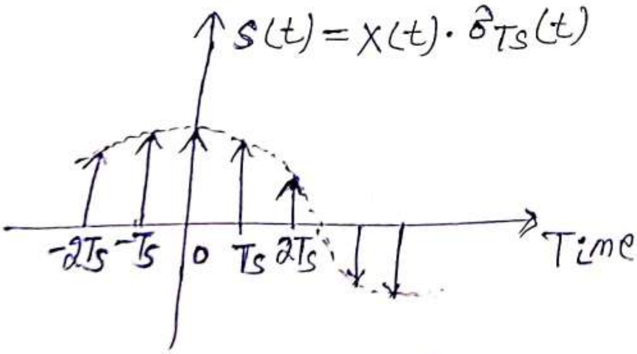
- It uses sample and hold circuit.
- It is practically possible like natural sampling but Flat top sampling is easier compared to natural sampling.
- It has very high noise interference.

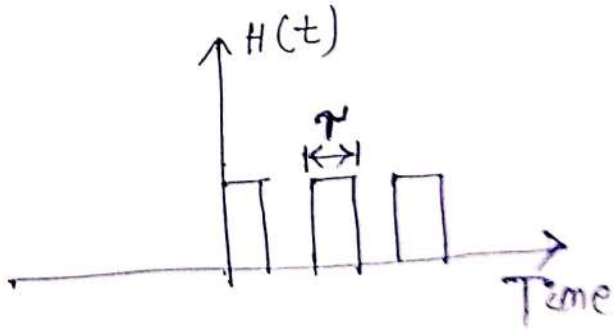


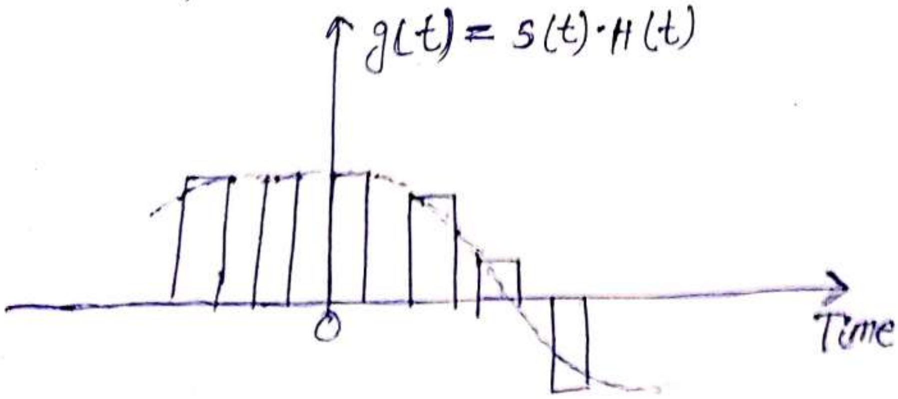




$$\delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$







τ = Time period that capacitor holds the output.

H(t) = Function of Discharge switch

s(t) signal is holded up to time period τ.

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↳ $g(t) = s(t) \cdot H(t)$

$$= \sum_{n=-\infty}^{\infty} x(t) \cdot h(t-nT_s)$$

In Frequency Domain $G(\omega) = f_s \cdot \sum_{n=-\infty}^{\infty} X(F-nF_s) \cdot H(F)$

↳ By sampling switch, sampling can be done. Discharge switch will define the time period up to which capacitor will charge. By pressing on Discharge switch, capacitor will discharge. By pressing on to sampling switch, then sampled output obtained that is constant voltage which is similar across capacitor. So, sampling switch is identical to impulse train switch.

Sampling switch on → capacitor will charge

Discharge switch on → capacitor will discharge, and output will zero.

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Performance comparison of sampling Techniques :-

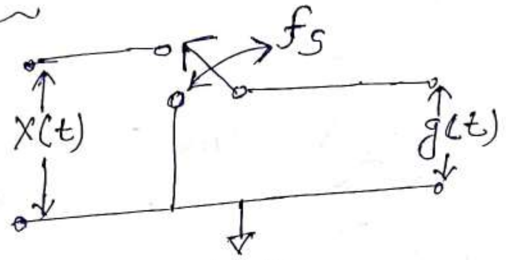
(a) performance parameters :

(1) sampling principle :-

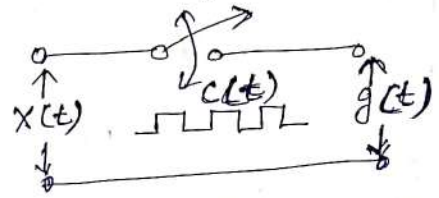
- In Ideal sampling, Multiplication is done.
- In Natural sampling, chopping is done.
- In Flat Top sampling, sample and Hold circuit is used.

(2) Generation circuit :-

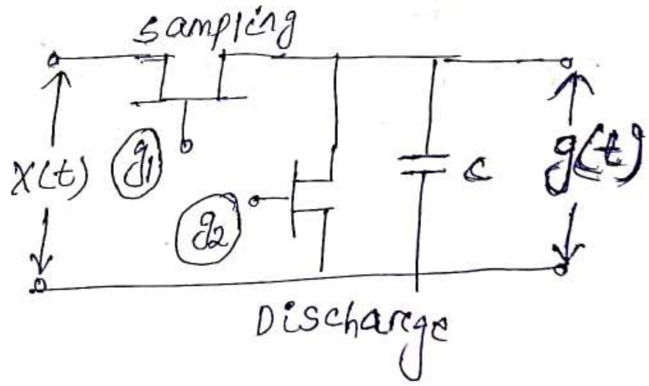
- In Ideal sampling,



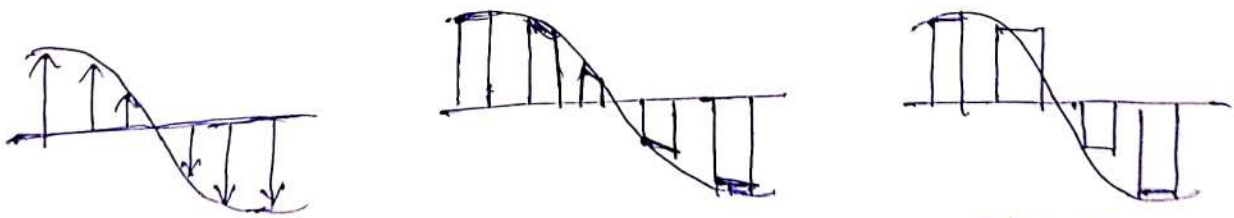
- In Natural sampling;



- In Flat top sampling,



(3) Waveforms :-



Ideal Sampling   Natural Sampling   Flat Top Sampling

(4) Feasibility :-

- Ideally Sampling practically not possible.
- Natural sampling practically used.
- Flat Top sampling practically used.

(4) Noise Interference :-

↳ In Ideal sampling Noise interference is very high, In Natural sampling it is Less, and in Flat Top Sampling Noise Interference is high.

(5) Time Domain Representation :-

$$g(t) = \sum_{n=-\infty}^{\infty} x(t) \cdot \delta(t - nT_s)$$ [For Ideal sampling]

$$g(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} x(t) \cdot \text{sinc}(nf_s \tau) \cdot e^{j2\pi f_s t}$$ [For Natural sampling]

$$g(t) = \sum_{n=-\infty}^{\infty} x(t) \cdot h(t - nT_s)$$ [For Flat Top sampling.]

(6) Frequency Domain Representation :-

$$G(f) = f_s \cdot \sum_{n=-\infty}^{\infty} x(f - nf_s)$$ [For Ideal sampling.]

$$G(f) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}(nf_s \tau) \cdot X(f - nf_s)$$ [For Natural sampling.]

$$G(f) = f_s \cdot \sum_{n=-\infty}^{\infty} X(f - nf_s) \cdot H(f)$$ [For Flat Top sampling.]

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BIT RATE AND BAUD RATE :-

↳ Bit Rate : It is number of Bits per second (bits/sec)
(R_b) or R

↳ Baud Rate : It is number of symbols per second or Elements per second. $\left[\frac{\text{symbols}}{\text{second}} / \frac{\text{elements}}{\text{second}}\right]$

(r) ~ notation of Baud rate

↳ If n = number of bits/symbol (or) bits/element

Then $\boxed{r = \frac{R}{n}}$

↳ Baud rate always Less than Bit Rate.

$\boxed{r < R}$

↳ Total number of symbols [elements] $= L = 2^n$

Example :- An Analog signal carries 4 bits/signal elements. If 1000 signal elements are sent per second. Then Find the Bit Rate.

sol^n :- $n = 4$ bits/elements , Baud rate $r = 1000$ baud / $\frac{\text{elements}}{\text{second}}$ / $\frac{\text{symbols}}{\text{second}}$

$R = n \cdot r = 4 \times 1000 = 4000$ bits/sec
$= \boxed{4 \text{ Kbps}}$

Total number of elements or symbols $L = 2^n = 2^4 = \boxed{16}$

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Que-2 :- An Analog signal has a bit rate of 8000 bps and a baud rate of 1000 baud. How many data elements are carried by each signal element? How many signal elements do we need?

sol^n :- $R = 8000$ bps , $r = 1000$ baud ; $n = ?$, $L = ?$

$n = \frac{R}{r} = \frac{8000}{1000} = 8$ bits/element

Total number elements $L = 2^n = 2^8 = 256$

QUANTIZATION :-

↳ A continuous signal such as voice has a continuous range of Amplitudes therefore its samples have a continuous amplitude range.

↳ In otherwords we can say, within the finite Amplitude range of signal we can find an infinite number of Amplitude levels.

↳ It is not necessary to transmit the exact amplitude of the samples because any human sense (the ear or the eye) works as an ultimate receiver that can detect finite intensity differences.

↳ This means that the original continuous signal may be approximated by a signal constructed of discrete amplitudes selected on a minimum error basis from an available set.

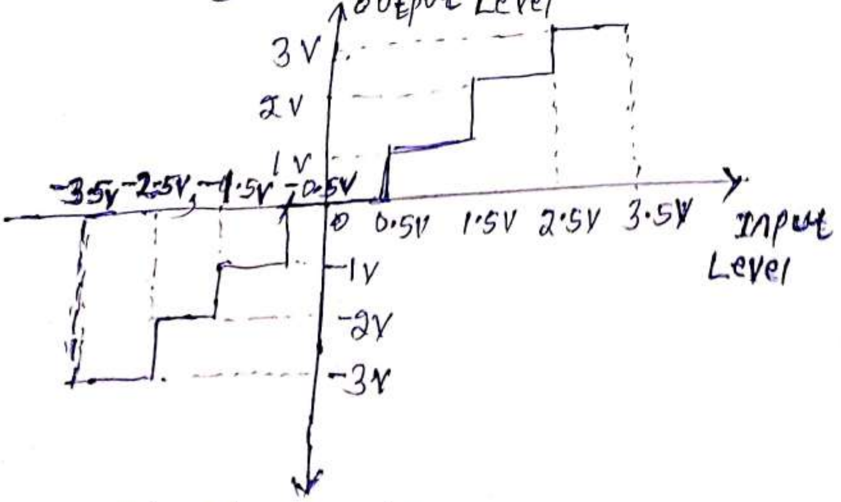
↳ Amplitude Quantization is defined as the process of transforming the sample amplitude $m(nT_s)$ of a message signal $m(t)$ at time $t = nT_s$ into a discrete amplitude $m_q(nT_s)$ taken from a finite set of possible Amplitudes.

↳ Quantizer can be of a uniform or non-uniform type.

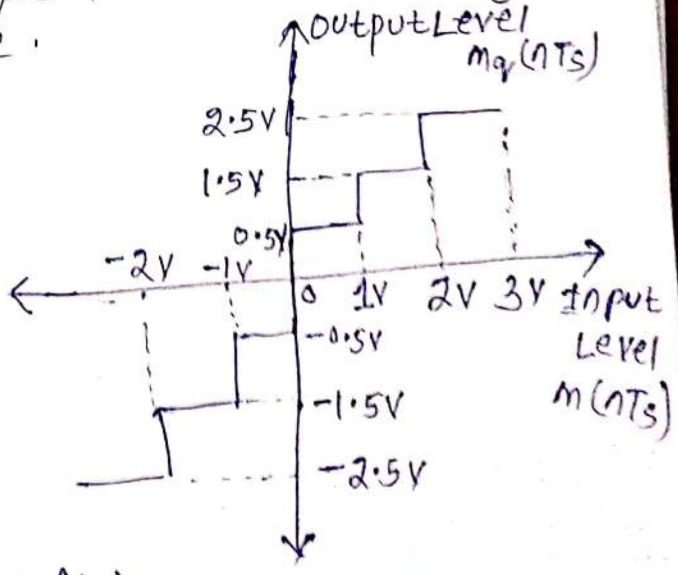
↳ In a uniform quantizer, the representation levels are uniformly spaced otherwise the quantizer is non uniform.

↳ The quantizer characteristics can be of two types

(a) Mid tread (b) Midrise.



(a) Midtread Type



(b) Midrise Type

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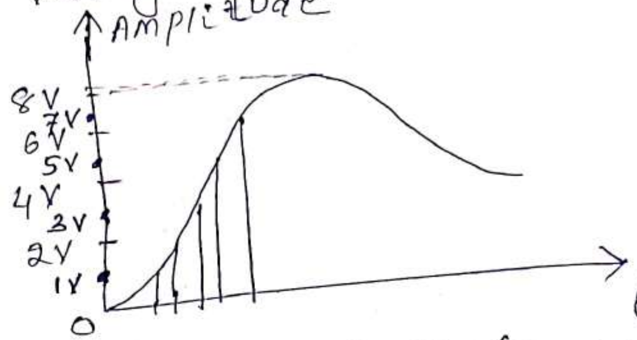
QUANTIZATION PROCESS :-

↳ Total dynamic range of the signal is divided in to L equal number of steps.

↳ Middle of each step will be selected as quantization voltage. ↳ Each of voltage corresponding to a step will be rounded off to middle of the step (or) each of the sample will be rounded off to one of the nearest quantization voltage.



Take $L = 4$

1 V → 00 [Encoded the quantization voltage.]
3 V → 01
5 V → 10
7 V → 11

Sample voltage	Quantization voltage	Encoded output	Quantization Error Q_e (Sampled voltage − quantized voltage)
0.6 Volt	1 Volt	00	−0.4 V
1.7 Volt	1 Volt	00	0.7 V
2.2 V	3 V	01	−0.8 V
0 V	1 V	00	−1 V
8 V	7 V	11	1 V

$$\text{Step size } \Delta = \frac{V_{max} \sim V_{min}}{\text{Number of Levels}} = \frac{8 \sim 0}{4} = 2$$

$Q_e(max)$ = Maximum Quantization Error $= \pm \frac{\Delta}{2}$

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NONUNIFORM QUANTIZATION :-

↳ If quantization characteristics is nonlinear then step-size is not constant, it means quantization is non-uniform quantization.

↳ In non uniform quantization, stepsize reduce with respect to reduction. in signal so quantization noise decreases. ↳ By companding we can achieve it. ↳ Non uniform quantization is generally used for speech and music signals.

↳ crest factor = $\frac{\text{peak value of signal}}{\text{RMS value of signal}} = \frac{X_{max}}{X_{rms}}$

↳ crest factor usually very high for speech and music signals.

↳ signal power $p = \frac{x^2(t)}{R}$, where $x(t)$ = Mean value of signal

$R = 1$ for normalized power

so, power $\boxed{p = x^2(t)}$

↳ crest factor $C.F. = \frac{X_{max}}{X_{rms}} = \frac{X_{max}}{\sqrt{x^2(t)}} = \frac{X_{max}}{\sqrt{P}}$

↳ For normalized signal $X_{max} = 1$

$C.F. = \frac{1}{\sqrt{P}} \Rightarrow P = \frac{1}{(C.F.)^2}$

↳ For Non sinusoidal signal, signal to Noise Ratio

$SNR = 3 \times 2^N \times P$ where P = power

↳ For voice & speech signal $CF >> 1$, so $P << 1$

Hence SNR is poor

↳ By using Non uniform quantization, we can change the step size with respect to signal. For weak signal we decrease the step size and for strong signal we increase the step size. That will improve SNR.

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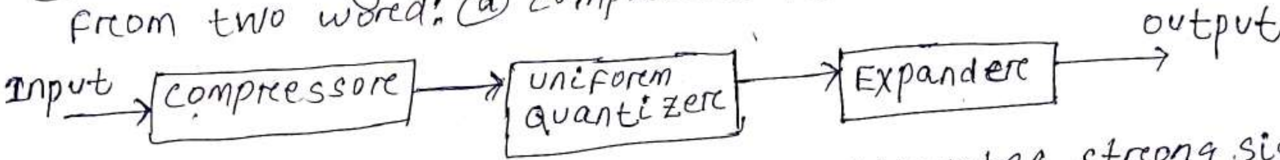
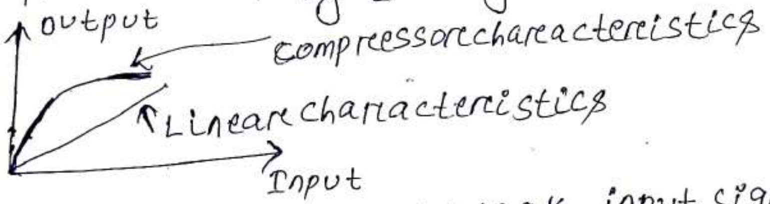
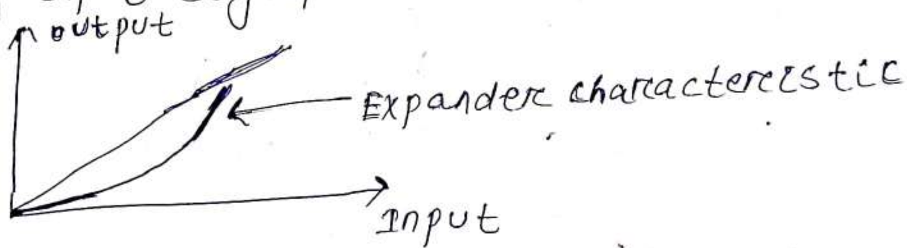
COMPANDING :- (i) companding is Nonuniform Quantization.
(2) It is required to be implemented to improve SNR of weak signal. (3) Quantization Noise is given by $\boxed{N_q = \frac{\Delta^2}{12}}$ for uniform quantization. and N_q is very high of weak signals in uniform quantization. In uniform quantization step size Δ is constant.
(4) For weak signal noise is constant. (5) companding is derived from two word: (a) compression, (b) Expansion.

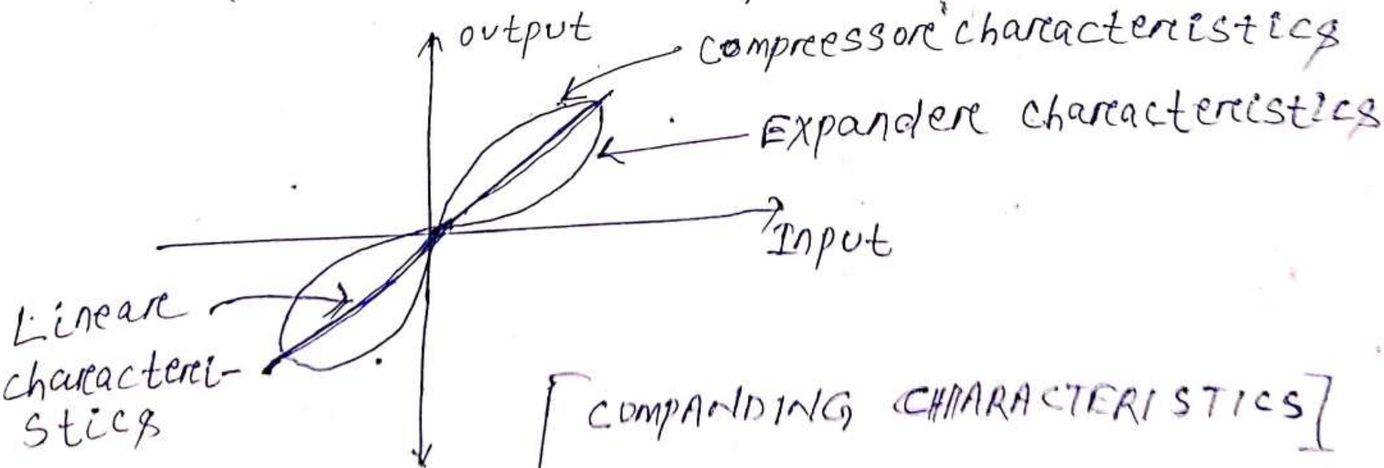


↳ compressor amplify low signal and attenuates strong signal.



↳ Expander attenuates weak input signal, and amplify the strong input signal





[COMPANDING CHARACTERISTICS]

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μ-LAW COMPANDING FOR Non UNIFORM QUANTIZATION :-

↳ It is very populare in USA and Japan. ↳ Input, output relationship is given by $\frac{|y|}{X_{max}} = \frac{\ln\left[1 + \mu \cdot \frac{|x|}{X_{max}}\right]}{\ln[1+\mu]}$

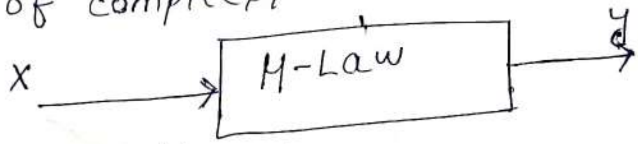
where,

x = Amplitude of Input signal at a particular Instant.

y = compressed output signal.

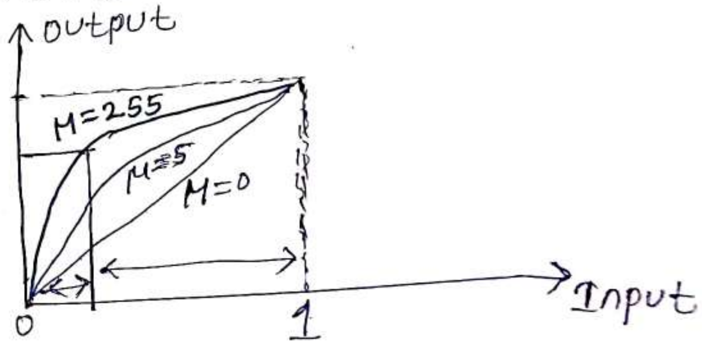
X_{max} = Maximum Amplitude of Input signal

μ = unitless parameter used to define the amount of compression.



$\left(\frac{|X|}{X_{max}}\right)$ → Range is (0 to 1)

↳ For $\mu = 0$, $\frac{|y|}{X_{max}} = \frac{\ln[1+0]}{\ln[1+0]} = 1$, so There is no compression.



↳ Larger the value of μ, results in to larger compression of output to input with higher amplitude.

↳ Recently in digital Transmission we use 8 bit PCM with $\mu = 255$.

x x —— x —— x ——

A-Law companding for Non uniform quantization :-

↳ It is very popular in India and in many European countries.

↳ It has slightly Flatter output characteristics than μ-Law.

↳ Input to output relationship is given by

$$\frac{|y|}{X_{max}} = \begin{cases} \dfrac{A\frac{|x|}{X_{max}}}{1+\ln A} & ; \; 0 \le \dfrac{|x|}{X_{max}} \le \dfrac{1}{A} \\[2ex] \dfrac{1+\ln\left[\frac{A|x|}{X_{max}}\right]}{1+\ln A} & ; \; \dfrac{1}{A} \le \dfrac{|x|}{X_{max}} \le 1 \end{cases}$$

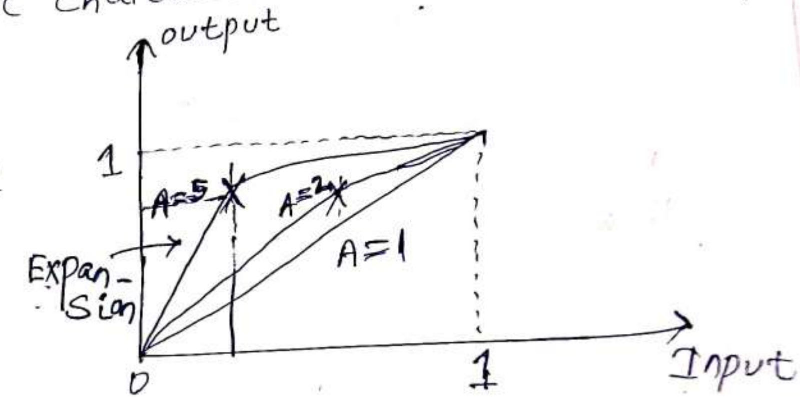
$$x \longrightarrow \boxed{\text{A Law}} \longrightarrow y$$

$\left(\frac{|x|}{X_{max}}\right) \rightarrow$ Range is 0 to 1.

↳ If $A=1$, $\frac{|y|}{X_{max}} = \frac{A|x|/X_{max}}{1+\ln A} = \frac{|x|}{X_{max}}$, That means output = Input. So There is no compression.

↳ Larger the value of A, Larger the compression. Larger the value of A, linear characteristics shift towards Left.

↳ In India for digital Telephone system we use $A=87.6$



— x — x —

1

INTRODUCTION TO DIGITAL SIGNAL PROCESSING

1.1 Introduction

Digital Signal Processing (DSP) is an area of science and technology that has developed rapidly over the past few decades. The techniques and applications of DSP are as old as Newton and Gauss and as new as Digital Computers and Integrated circuits (ICs). The rapid development of DSP is a result of the significant advances in Digital Computer technology and IC fabrication.

DSP is concerned with the representation of signals by sequences of numbers or symbols and processing of these sequences. Processing means modification of sequences into a form which is in some sense more desirable.

In another words, DSP is a mathematical manipulation of discrete-time signals to get more desirable properties of the signal, such as less noise or distortion.

The classical numerical analysis formulae such as those used for interpolation, differentiation and integration are also DSP algorithms.

DSP finds application in various fields such as speech communication, data communication, image processing, radar engineering, seismology, sonar engineering, biomedical engineering, acoustics, nuclear science and many others.

DSP can be applied to one dimensional signals as well as multidimensional signals. Example of one dimensional signal is speech and example of two-dimensional signal is image. Many picture processing applications require the use of two dimensional signal processing techniques. Two-dimensional signal processing includes X-ray enhancement, analysis of aerial photographs (these photographs are necessary for detection of forest fire or crop damage), analysis of satellite weather photographs etc. Analysis of seismic data is required in oil exploration, earth quake measurements and monitoring of nuclear tests. These utilize multidimensional signal processing techniques. The impact of DSP techniques will undoubtedly promote revolutionary advances in many fields of application. A notable example is telephony where digital techniques dramatically increased economy and flexibility in implementing switching and transmission systems.

1.2 Signal Processing Systems

A system responds to particular signals by producing other signals having some desired behaviour.

Signal processing systems are of two types depending on the type of signal to be processed.

1. Continuous-time Systems.
2. Discrete-time Systems.

1.2.1 Continuous-time Systems

Continuous-time systems are the systems for which both input and output are continuous-time signals. H(s) is the transfer function of a continuous-time system. Fig. 1.1 illustrates the block diagram of a continuous-time system.

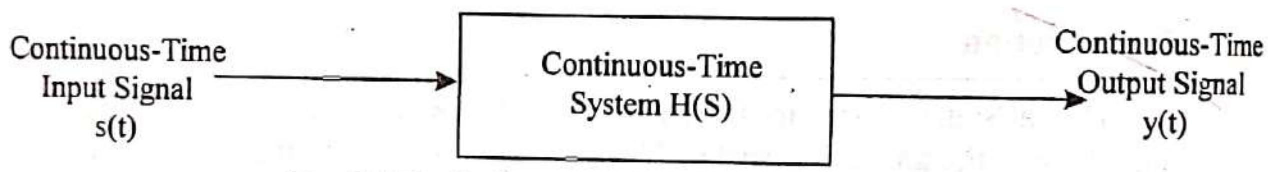


Fig. 1.1 Block diagram of continuous-time system.

An example of continuous-time system is an analog filter which is used to reduce the noise corrupting a message signal.

1.2.2 Discrete-time Systems

Discrete-time systems are systems for which both the input and output are discrete-time signals. H(z) is the transfer function of a discrete-time system. Fig. 1.2 illustrates the block diagram of a discrete-time system.

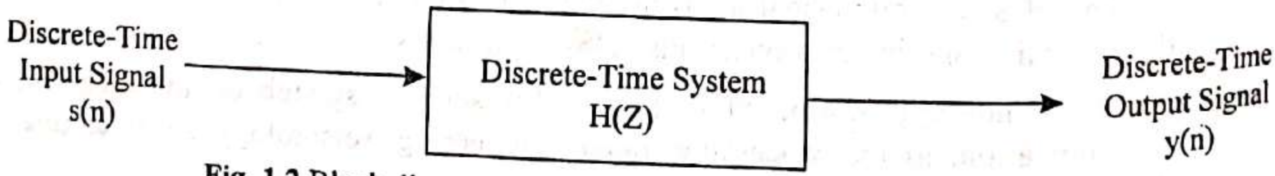


Fig. 1.2 Block diagram of discrete-time system. An example of a discrete-time system is a digital computer.

1.3 Signal Processing

Changing the basic nature of signal to obtain the desired shaping of the input signal is called signal processing. Signal processing is concerned with the representation, transformation, and manipulation of signals and the information they contain.

Signal processing is of two types depending upon the type of signal to be processed.

1. Analog Signal Processing (ASP).
2. Digital Signal Processing (DSP)

1.3.1 Analog Signal Processing

In analog signal processing, continuous-amplitude continuous-time signals are processed. Various types of analog signals are processed through low pass filters, high pass filters, band pass filters and band reject filters to obtain the desired shaping of the input-signal. Another example of analog signal processing is the production of modulated carrier using High Frequency (HF) oscillator, and the modulating audio signal and a modulator. Fig. 1.3 illustrates the block diagram of an ASP system.

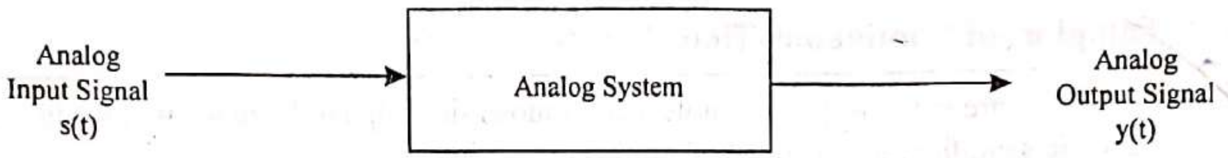


Fig. 1.3 Block diagram of ASP system.

1.3.2 Digital Signal Processing

Digital signal processing (DSP) is a numerical processing of signals on a digital computer or some other data processing machine. Fig. 1.4 illustrates the block diagram of DSP system.

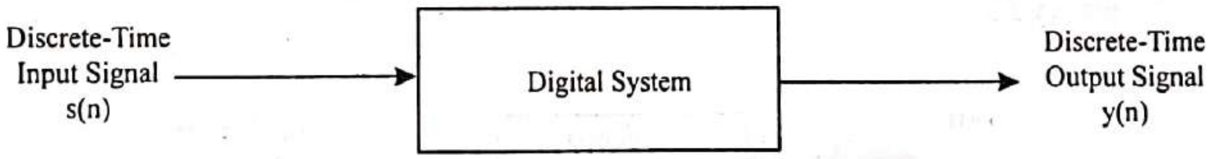


Fig. 1.4 Block diagram of DSP system.

A digital system such as digital computer takes input signal in discrete-time sequence form and converts it in discrete-time output sequence.

1.4 Elements of digital signal processing system

1. A signal is a physical quantity that varies with time, space, or any other independent variable.
2. A system is defined as a physical device that performs an operation on a signal.
3 Signal processing is any operation that changes the characteristics of a signal. These characteristics include the amplitude, shape, phase & frequency content of the signal.
4. The DSP is a numerical processing of signals on a digital computer or some other data processing machine.
5. The block diagram of DSP system is,

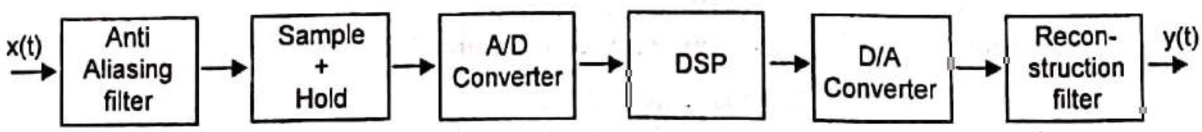


Fig. 1.5

6. The input signal is applied to the anti-aliasing Filter. The low pass filter removes the high frequency noise and to band-limit the signal.
7. The sample & hold provides the discrete time signal to A/D converter.
8. The ADC converts analog signal to digital signal.
9. The DSP may be a large programmable digital computer programmed to perform, the desired operation on the input signal.
10. The output of DSP is converted to analog signal by DAC.
11. The high frequency components in DAC output is released by the reconstruction filter.

1.5 Sampling of Continuous-Time Signals

There are many ways to sample a continuous-time signal. Here we will discuss only periodic sampling. It is also called uniform sampling.

If $s_a(t)$ is a continuous-time signal. Periodical measurement of continuous-time signal is called periodic sampling or uniform sampling.

By periodic sampling of continuous-time signal, we can get discrete-time signal.

Discrete-time signal, $s_a(nT_s) \equiv s_a(t)|_{t=nT_s}$

where T is the sampling period and reciprocal of sampling period is termed as sampling frequency F_s.

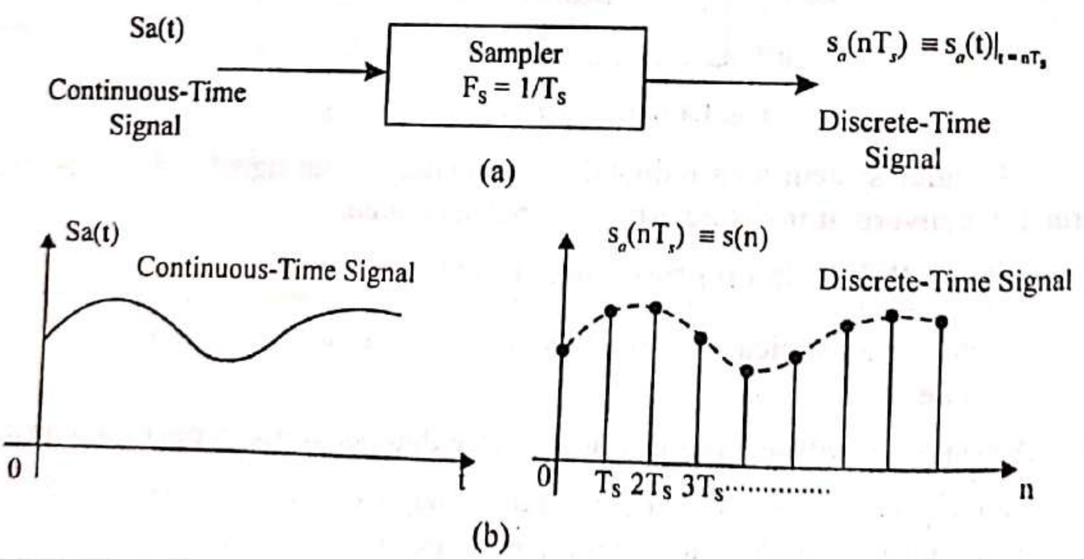


Fig. 1.6 (a) Block diagram of a sampler, (b) Periodic sampling of continuous-time signal.

1.5.1 Nyquist Rate

Nyquist rate is defined as minimum sampling rate required for perfect reconstruction of sampled signal at the receiver.

If any signal has highest frequency component F_{max}, then

$$\text{Nyquist rate} = 2 \times F_{max}$$

1.5.2 Sampling Theorem

It is stated as : For perfect reconstruction of sampled signal at receiver, sampling rate or sampling frequency should be greater than or equal to Nyquist rate of the message signal.

According to the sampling theorem,

Sampling rate $\geq$ Nyquist rate, $2F_{max}$

Periodic sampling establishes a relationship between the time variables t and n of continuous-time and discrete-time signals, respectively.

Consider a continuous-time signal, $s_a(t) = A_s \cos(2\pi F_{max} t + \theta)$

Sampling periodically at a sampling rate $F_s = 1/T_S$ samples per second produces

$$s(n) = s_a(nT_s) = A_s \cos(2\pi F_{max} nTs + \theta)$$

$$= A_s \cos\left(2\pi F_{max} n \frac{1}{F_s} + \theta\right)$$

$$= A_s \cos\left(2\pi \frac{F_{max}}{F_s} n + \theta\right)$$

$$= A_s \cos(2\pi f n + \theta), -\infty < n < \infty$$

where $f = \frac{F_{max}}{F_s}$ is the frequency variable for discrete-time signals

F_{max} is the frequency variable for continuous-time signals

F_s is the sampling rate

1.5.3 Aliasing

When sampling frequency is less than Nyquist rate then aliasing phenomenon occurs

Nyquist rate = $2F_{max}$ = 2 × Highest frequency component of message signal

If sampling rate < Nyquist rate than it is called under sampling and in this case aliasing phenomenon occurs.

If sampling rate > Nyquist rate then it is called over sampling and in this case no aliasing phenomenon occurs. Infact this is a suitable and necessary condition for sampling process.

Aliasing phenomenon is defined as a phenomenon of high frequency component in a spectrum of a signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version.

Fig. 1.7 shows spectra of signals showing the sampling relations between analog and digital systems for a properly sampled input signal.

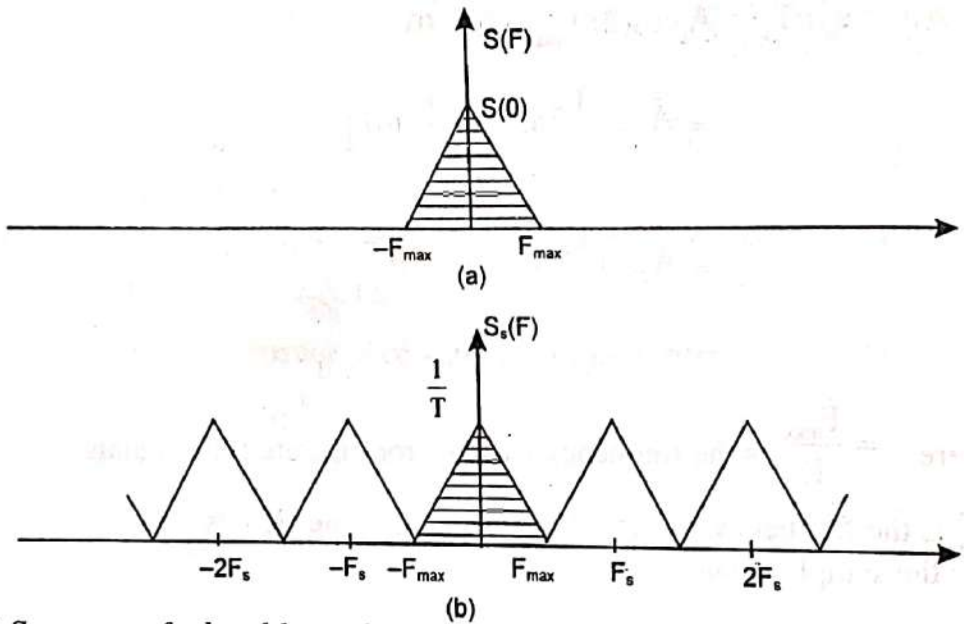
Fig. 1.8 shows the effect of under sampling on the digital frequency response.

Aliasing problem occurs when sampling frequency $F_s < 2F_{max}$. In this case sampling frequency F_s *is* not sufficiently high to prevent the shifting of high frequency information into lower frequencies. Such transference of information from one band of frequencies to another is called Aliasing and the resulting frequency response is called an aliased representation of the original signal.

There are two corrective measures which are used to eliminate aliasing

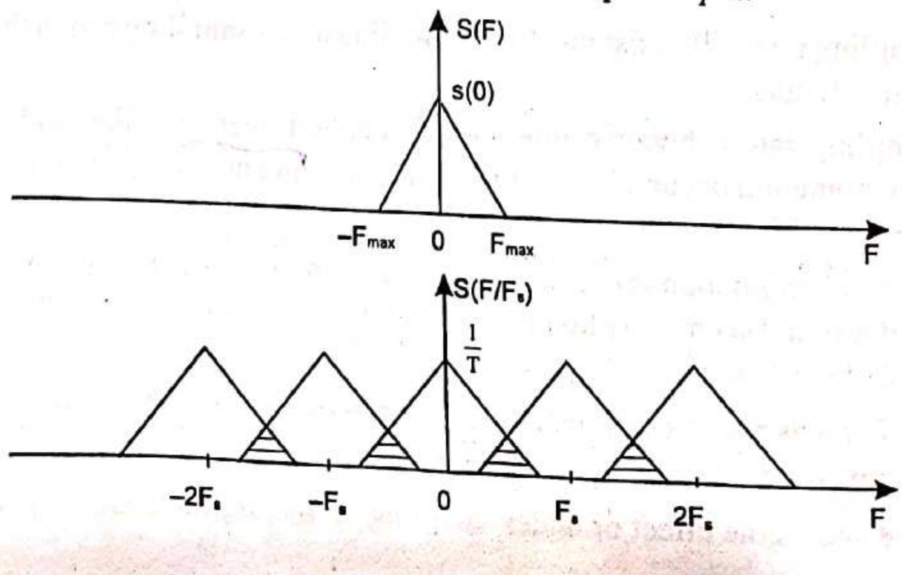
1. a pre-alias low pass filter is used before sampling for attenuating those high frequencies that are not essential for the transmission of information.

2. a pre-alias low pass filtered signal is sampled at a rate slightly higher than the Nyquist rate ($F_s > 2F_{max}$).



(a) *Spectrum of a band-limited analog signal s(t). (b) Spectrum of a sampled version of signal s(t) for a sampling frequency* $F_s = 2F_{max}$.

Fig. 1.7 *Spectrum of signals showing the sampling relations between analog and digital systems for a properly sampled input.*



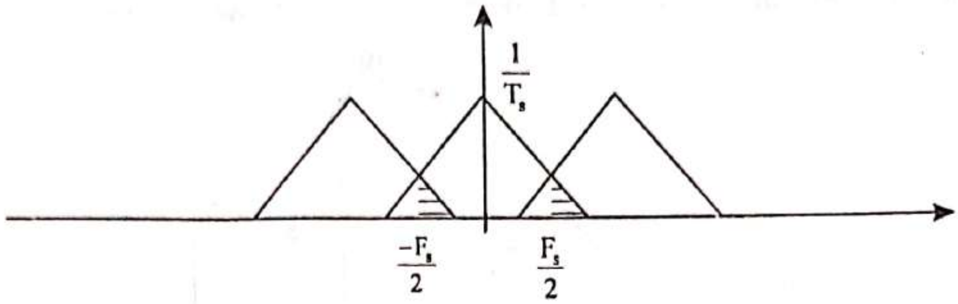


Fig. 1.8 *The effect of under sampling an analog signal on its digital frequency response showing aliasing around the folding frequency $F_s/2$.*

1.5.4 Anti-Aliasing Filter

In practice, communication signals have frequency spectra consisting of low frequency components as well as high-frequency noise components. If we select sampling frequency F, all signals with frequency higher than $\frac{\Omega_s}{2}$ appear as signals of frequencies between 0 and $\frac{\Omega_s}{2}$ due to aliasing effect. To avoid aliasing we can choose very high sampling frequency. But sampling at very high frequencies introduces numerical errors. Therefore, to avoid aliasing errors caused by the undesired high frequency signals, an analog lowpass filter, called an anti-aliasing filter is used prior to sampler (refer Fig. 1.2) to filter high frequency components before the signal is sampled.

1.5.5 Sample-and-hold circuit

The output of the anti-aliasing filter is fed to a sample-and-hold (S/H) circuit. It samples the analog input signal at uniform intervals and holds the sampled value constant as long as the A/D converter takes time for accurate conversion. The use of sample-and-hold circuit allow the ADC to operate slowly.

The basic circuit diagram of sample-and-hold circuit is shown in Fig. 1.9.

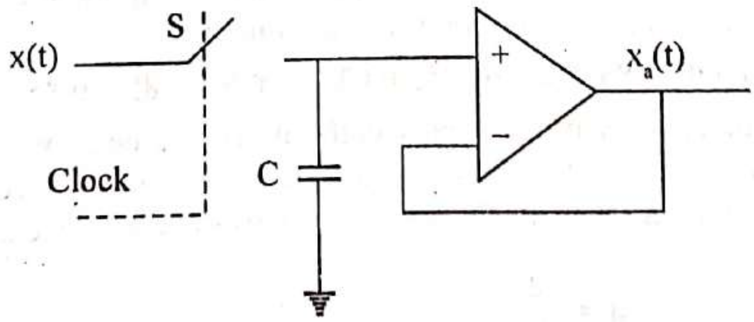


Fig. 1.9 *Sample-and-hold circuit*

During sample mode the switch S is closed allowing the capacitor C to charge to input voltage. During the hold period the switch remains open, the charge on the capacitor holds the voltage across it. A digital clock controls the switching operation. The voltage follower acts as a buffer between the capacitor and the input stage of the A/D converter. The input and output waveforms of a sample-and-hold circuit is shown in Fig. 1.10.

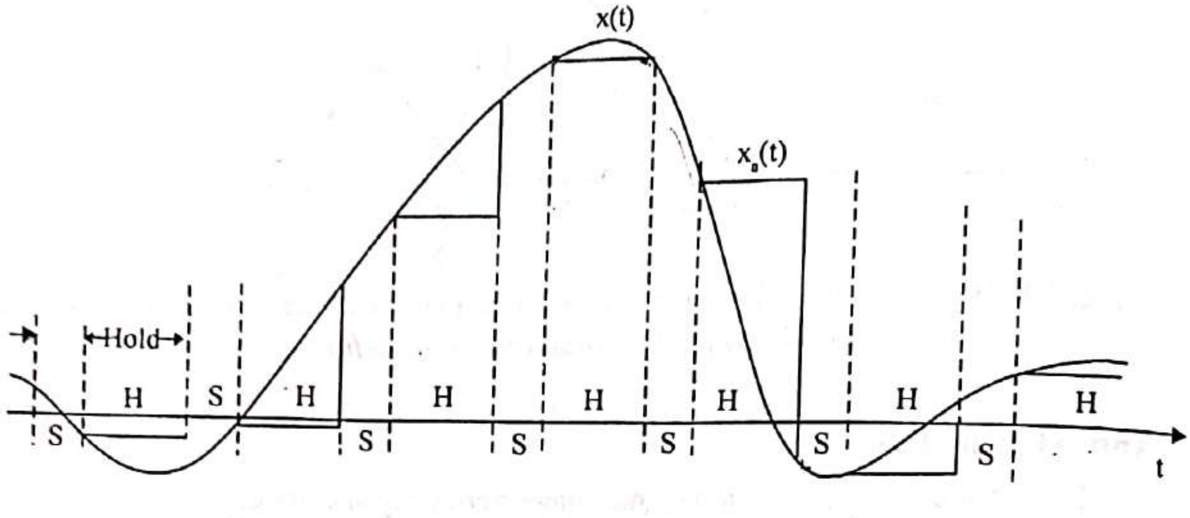


Fig. 1.10 Input and output waveforms of S/H circuit

1.5.6 Quantization

The process of converting a discrete-time continuous amplitude signal x(n) into a discrete-time discrete amplitude signal $x_q(n)$ is known as quantization. This is done by rounding off each sample in x(n) to nearest quantization level. Then each sample in $x_q(n)$ is represented by a finite number of digits using a coder. If a signal with amplitude range R is represented by an $b+1$ bit word (including sign bit) then the number of values, or quantization levels, that can be represented is 2^{b+1}. The difference between adjacent levels, or the quantization step interms of the range of the signal is

$$q = \frac{\text{range of signal}}{\text{Number of quantization levels}} = \frac{R}{2^{b+1}}$$

With fixed point representation of fractional number, if the range of the signal exceeds ± 1, it is necessary to scale the signal.

The process of quantization is shown in Fig. 1.11. The time axis of the discrete-time signal is labelled with sample number (n = 0, 1,2 ....).Corresponding to different values of sample number n, the discrete time continuous amplitude signal is shown in Fig. 1.11. We can represent the sample values by a sequence

$$x(n) = \{0, 0.620, 0.85, 0.85, 0.575, -0.03, -0.625, -0.85, -0.85, -0.575, 0\}$$

Let a b + 1 bit ADC is used to represent the above sequence. With b + 1 binary digits 2^{b+1} quantization levels can be obtained and the input can be resolved to one part in 2^{b+1}. If the input signal has a range of 2V, then the quantization step size is equal to

$$q = \frac{2}{2^{b+1}} = 2^{-b}$$

If b + 1 is equal to 4, the quantization step size is equal to 0.125. Thus the input signal must change atleast 0.125 in order to produce a change in the output.

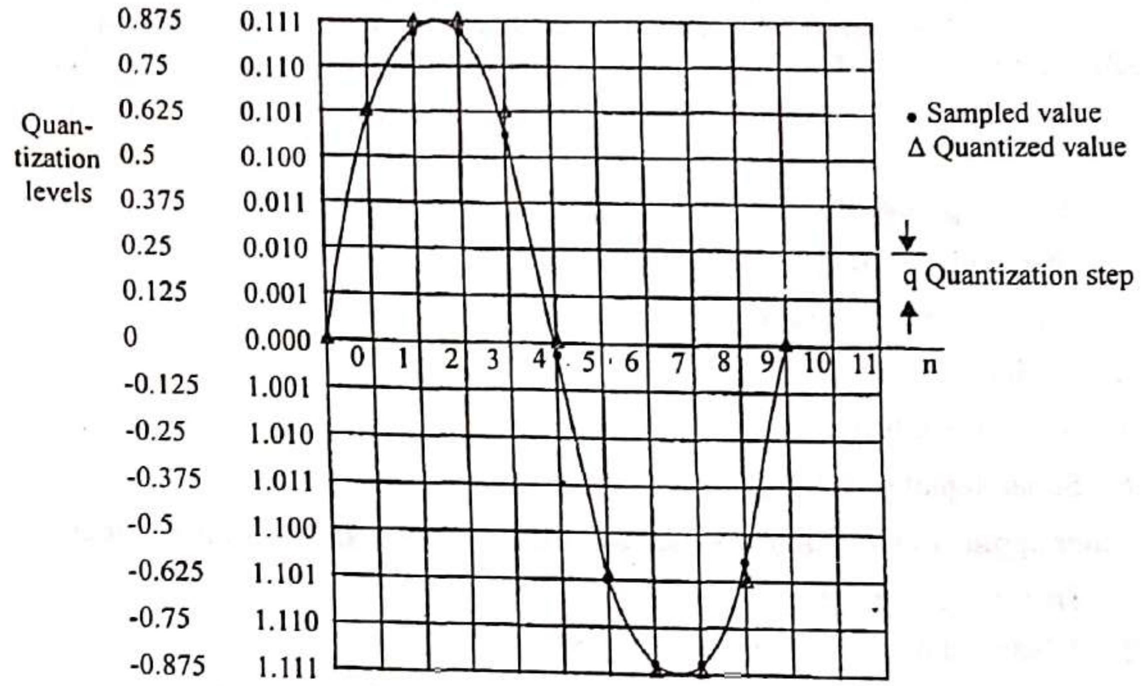


Fig. 1.11 quantization of Signal

The process of converting x(n) to finite number of digits introduces an error known as quantization noise. It is a sequence e(n) defined as the difference between the quantized value and the actual sample value. Thus $e(n) = x_q(n) - x(n)$

Table 1.1 Illustration *of* quantization using rounding

n	Sampled value x(n)	binary representation	Rounding	Quantized value	quantization noise $e(n) = x_q(n) - x(n)$
0	0	0.00000000	0.000	0	0
1	0.620	0.10011110	0,101	0.625	0.005
2	0.85	0.11011001	0.111	0.875	0.025
3	0.85	0.11011001	0.111	0.875	0.025
4	0.575	0.10010011	0.101	0.625	0.05
5	-0.03	1.00000111	1.000	0	0.03
6	- 0.625	1.10100000	1.101	- 0.625	0
7	-0.85	1.11011001	1.111	-0.875	- 0.025
8	-0.85	1.11011001	1.111	-0.875	- 0.025
9	-0.575	1.10010011	1.101	- 0.625	-0.05
10	0	0.00000000	0.000	0	0

1.6 Applications of Digital Signal Processing (DSP)

As a matter of fact, there are various application areas of digital signal processing (DSP) due to the availability of high resolution spectral analysis. It requires high speed processor to implement the Fast Fourier Transform (FFT). Some of these areas are can be listed as under :

1. Speech processing.
2. Image processing.
3. Radar signal processing.
4. Digital communications.
5. Spectral analysis.
6. Sonar signal processing.

Few other applications of digital signal processing (DSP) can be listed as under :

1. Transmission lines.
2. Advanced optical fibber communication.
3. Analysis of sound and vibration signals.
4. Implementation of speech recognition algorithms.
5. Very Large Scale Integration (VLSI) technology.
6. Telecommunication networks.
7. Microprocessor systems.
8. Satellite communications.
9. Telephony transmission.
10. Aviation.
11. Astronomy
12. Industrial noise control.

Now, let us discuss few major applications in brief:

1. Speech Processing

Speech is a one dimensional signal. Digital processing of speech is applied to a wide range of speech problems such as speech spectrum analysis, channel vocoders (voice coders) etc. DSP is applied to speech coding, speech enhancement, speech analysis and synthesis, speech recognition and speaker recognition.

2. Image Processing

Any two-dimensional pattern is called an image. Digital processing of images requires two-dimensional DSP tools such as Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT) algorithms and z-transforms. Processing of electrical signals extracted from images by digital techniques include image formation and recording, image compression, image restoration, image reconstruction and image enhancement.

3. Radar Signal Processing

Radar stands for "Radio Detection and Ranging". Improvement in signal processing is possible by digital technology. Development of DSP has led to greater sophistication of radar tracking algorithms. Radar systems consist of transmit-receive antenna, digital processing system and control unit.

4. Digital Communications

Application of DSP in digital communication specially telecommunications comprises of digital transmission using PCM, digital switching using Time Division Multiplexing (TDM), echo control and digital tape recorders. DSP in telecommunication systems are found to be cost effective due to availability of medium and large scale digital ICs. These ICs have desirable properties such as small size, low cost, low power, immunity to noise and reliability.

5. Spectral Analysis

Frequency-domain analysis is easily and effectively possible in digital signal processing using Fast Fourier Transform (FFT) algorithms. These algorithms reduce computational complexity and also reduce the computational time.

6. Sonar Signal Processing

Sonar stands for "Sound Navigation and Ranging". Sonar is used to determine the range, velocity and direction of targets that are remote from the observer. Sonar uses sound waves at lower frequencies to detect objects under water.

DSP can be used to process sonar signals, for the purpose of navigation and ranging.

1.7 Advantages of Digital Signal Processing (DSP) over Analog Signal Processing (ASP)

Digital Signal Processing (DSP) has following advantages over Analog Signal Processing (ASP) :

1. Digital signal processing operations can be changed by changing the program in digital programmable system. This means that these are flexible systems.
2. There is a better control of accuracy in digital systems compared to analog systems.
3. Digital signals are easily stored on magnetic media such as magnetic tape without loss of quality of reproduction of signal.
4. Digital signals can be processed of line, *i.e.*, these are easily transported.
5. Sophisticated signal processing algorithms can be implemented by DSP method.
6. Digital circuits are less sensitive to tolerances of component values.
7. Digital systems are independent of temperature, ageing and other external parameters.
8. Digital circuits can be reproduced easily in large quantities at comparatively lower cost.

9. Cost of processing per signal in DSP is reduced by time-sharing of given processor among a number of signals.
10. Processor characteristics during processing, as in adaptive filters can be easily adjusted in digital implementation.
11. Digital system can be cascaded without any loading problems.

1.8 Limitations of DSP

1. ***System complexity.*** System complexity increased in the digital processing of an analog signal because of the devices such as A/D and D/A converters and their associated filters.
2. ***Bandwidth limited by sampling rate.*** Band limited signals can be sampled without information loss if the sampling rate is more than twice the bandwidth. Therefore, the signals having extremely wide bandwidths require fast sampling rate A/D converters and fast digital signal processors. But there is practical limitation in the speed of operation of A/D converters and digital signal processors.
3. ***Power consumption.*** A variety of analog processing algorithms can be implemented using passive circuit employing inductors, capacitors and resistors that do not need any power, whereas a DSP chip containing over 4 lakh transistors dissipates more power (1 watt).

EXERCISE

1. What is a signal ? Give some example of signals.
2. Give the classification of signals.
3. What do you mean by signal processing ? Differentiate between analog signal processing and digital signal processing.
4. What are the basic elements of digital signal processing (DSP) system ?
5. List the advantages of digital signal processing over analog signal processing ?
6. Explain the importance of DSP in various fields of engineering and technology. Give a brief account of its applications.

❑❑❑

2

DISCRETE-TIME SIGNAL AND SYSTEMS

2.1 Introduction

In this modern age of microelectronics, signals and systems play very vital roles. It is an extraordinary subject with diverse applications in areas of science and technology such as circuit design, seismology, communications, biomedical engineering, energy generation and distribution, speech processing etc. Therefore, it is essential that every practising engineer and designer must have a thorough knowledge of this subject. Understanding of signals and systems is also must for study of other parts of engineering such as signal processing and control systems.

2.2 Signals

A signal may be a function of time, temperature, position, pressure, distance etc. Some signals in our daily life are music, speech, picture and video signals. Systematically, we can define a signal as *"A function of one or more independent variables which contains some information is called a signal'.*

In electrical sense, the signal can be voltage or current. The voltage or current is the function of time as an independent variable.

In daily life, we come across several electric signals such as Radio Signal, T.V. Signal, Computer Signal etc.

Many signals that we come across are naturally generated signals. However, few signals are also generated synthetically.

2.3 Discrete - Time Signals

Discrete-time signals are defined for discrete values of an independent variable (time). Discrete-time signal is not defined at instants between two successive samples.

Discrete-time signals are represented in two ways

$$s(n), \quad N_1 \leq n \leq N_2 \qquad ...(2.1)$$

where N_1 and N_2 are the first and the last sample point, respectively in a given discrete-time signal.

It represents non-uniformly spaced samples and these are shown in Fig. 2.1(a).

$$s(nT_s), \ N_1 < n < N_2 \qquad ...(2.2)$$

It represents uniformly spaced samples and these are shown in Fig. 2.1(b).

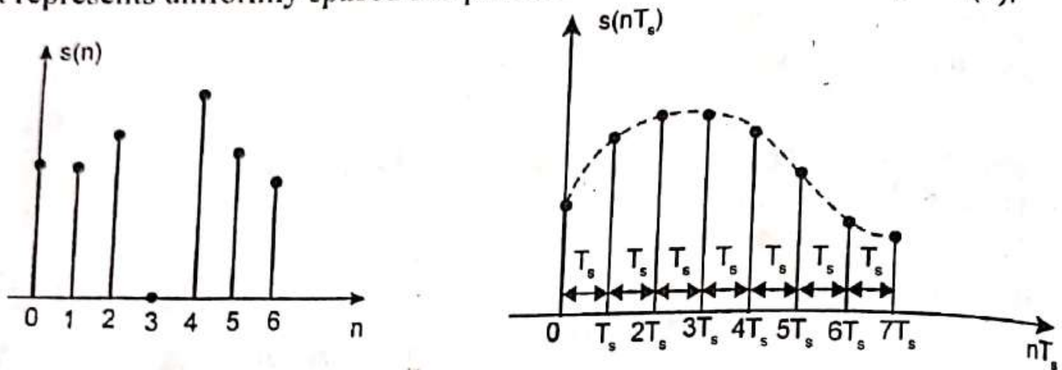


Fig. 2.1 (a) *Discrete-time signal showing non-uniformly spaced samples (there is no sampling period T_s) (b) Discrete-time signal showing uniformly spaced samples.*

2.3.1 Representation of Discrete-Time Signals

Discrete-time signal sequences can be represented in following four ways

1. Graphical Representation.
2. Functional Representation.
3. Tabular Representation.
4. Sequence Representation.

Graphical Representation. Discrete-time signals can be represented by a graph when the signal is defined for every integer value of n for $-\infty < n < \infty$. This is illustrated in Fig. 2.2.

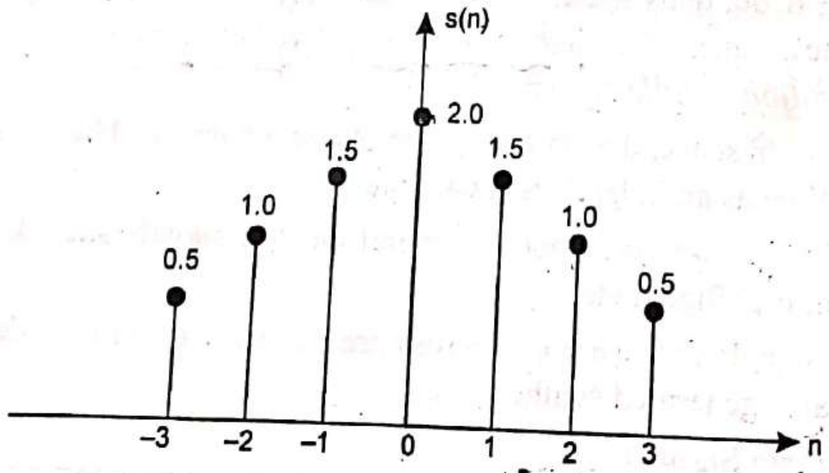


Fig. 2.2 *Graphical representation of a discrete-time signal.*

Functional Representation. Discrete-time signals can be represented functionally as given below

$$s(n) = \begin{cases} 2, \text{ for } n = 1,3 \\ 4, \text{ for } n = 2 \\ 0, \text{ elsewhere} \end{cases} \quad \text{...(2.3)}$$

Tabular Representation. Discrete-time signals can also be represented by a table as,

n	.....	−3	−2	−1	0	1	2	3	4	5	.....
s(n)		0	0	0	1	2	1	0	0	0	

Sequence Representation. An infinite-duration ($-\infty \le n \le \infty$) signal with the time as origin ($n = 0$) and indicated by the symbol ↑.

$$s(n) = \{\ldots 0, 0, 0, \underset{\uparrow}{1}, 3, 1, 0, 0\} \quad \text{...(2.4)}$$

2.3.2 Methods of Obtaining a Signal Sequence

There are three methods of obtaining a sequence :

1. To generate a set of numbers and order them into sequence form

   Example : $s(n) = n,\ 0 \le n \le N-1$ ...(2.5)

2. A sequence is generated by some recursion relation

   Example : $s(n) = \dfrac{1}{2} s(n-1)$ ...(2.6)

   with initial condition $s(0) = 1$
   generates a sequence

   $$s(n) = \left(\frac{1}{2}\right)^n, \ 0 \le n \le \infty \quad \text{...(2.7)}$$

3. A sequence is also obtained by periodic sampling of continuous-time signals. Periodic measurement of continuous-time signals is called periodic sampling.

   Discrete-time sequence, $s(nT_s) = s(t)|_{t=nT_s}, -\infty < n < \infty$ ...(2.8)

where T_s is the sampling interval and $s(t)$ is a continuous-time signal.

2.3.3 Some Elementary Discrete-Time Signals

There are some basic signals which play an important role in the study of discrete-time signals and systems.

These signals are given below

1. Unit-Sample (Impulse) Sequence, $\delta(n)$
2. Unit-Step Sequence, $u(n)$
3. Unit-ramp Sequence, $r(n)$
4. Exponential Sequence
5. Sinusoidal Sequence.

Unit-Sample Sequence. Fig. 2.3 shows a unit sample sequence, it is denoted by $\delta(n)$ and is defined as

impulse

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \qquad ...(2.9)$$

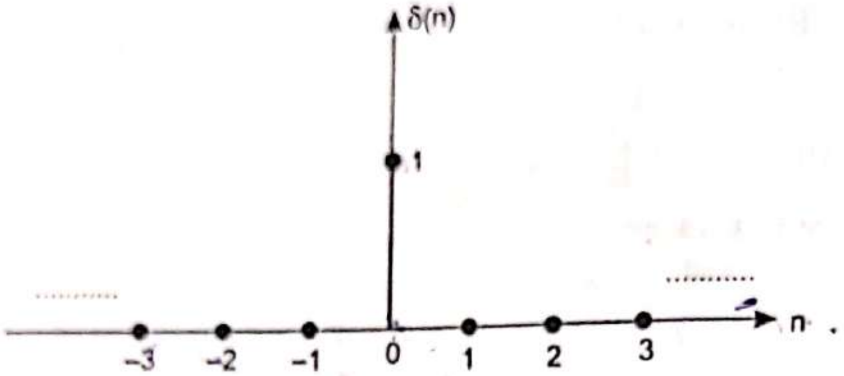


Fig. 2.3 *Graphical representation of* $\delta(n)$.

Unit-Step Sequence. It is denoted by $u(n)$ and is defined as $\delta(n)$.

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad ...(2.10)$$

Fig. 2.4 illustrates the graphical representation of unit-step sequence.

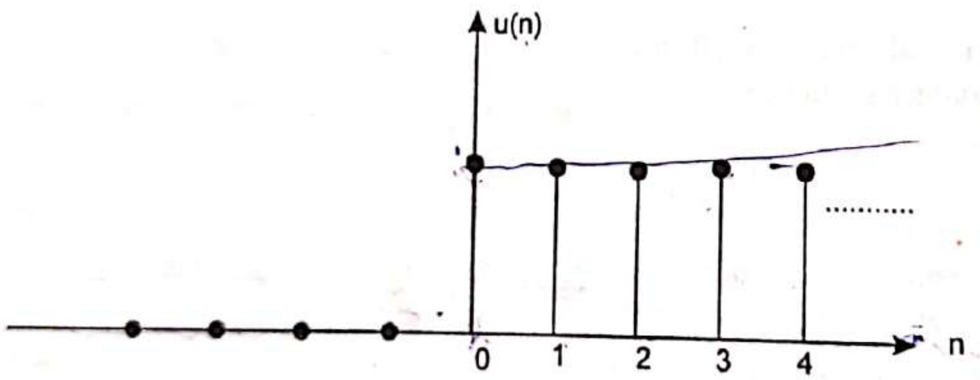


Fig. 2.4 *Graphical representation of* $u(n)$.

Unit-Ramp Sequence. It is denoted by $r(n)$ and is defined as

$$r(n) = \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \qquad ...(2.11)$$

Fig. 2.5 shows the graphical representation of unit-ramp sequence.

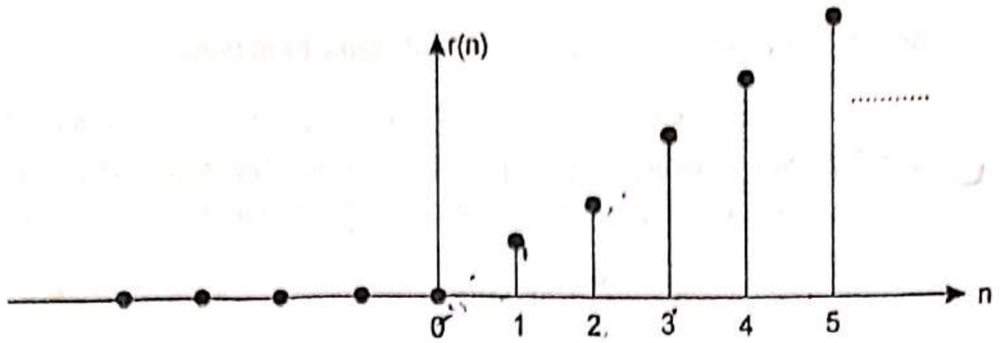


Fig. 2.5 *Graphical representation of r(n).*

Exponential Sequence. It is defined as

$$s(n) = (A)^n \text{ for all values of } n \qquad \text{...(2.12)}$$

If the parameter A is real, then $s(n)$ is a real sequence. Fig. 2.6 illustrates graphical representation of exponential sequence.

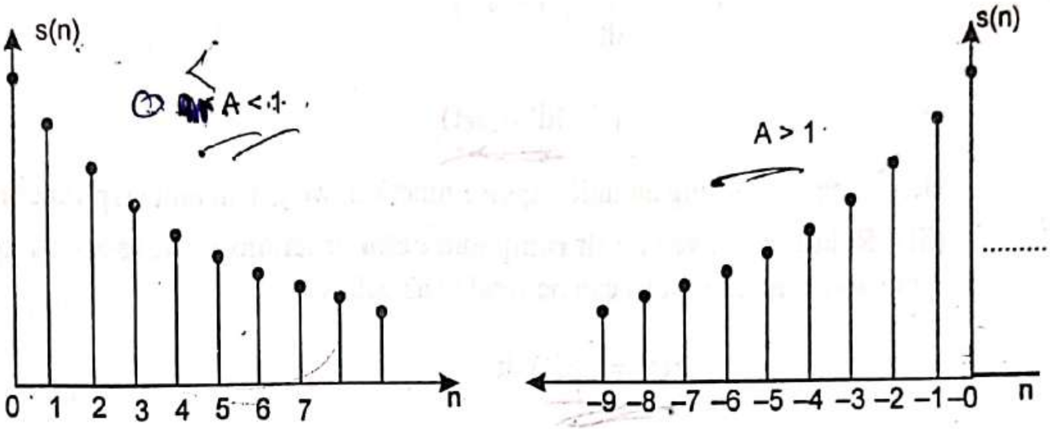


Fig. 2.6 *Graphical representation of exponential sequences.*

Sinusoidal Sequences. There are two types of sinusoidal sequences, one is called the sine sequence and the other is called cosine sequence.

Sine sequence is defined as

$$s(n) = \sin \omega_0 n, \text{ for all } n \qquad \text{...(2.13)}$$

and cosine sequence is defined as

$$s(n) = \cos \omega_0 n, \text{ for all } n \qquad \text{...(2.13)}$$

Fig. 2.7 illustrates the graphical representation of cosine type sinusoidal sequence.

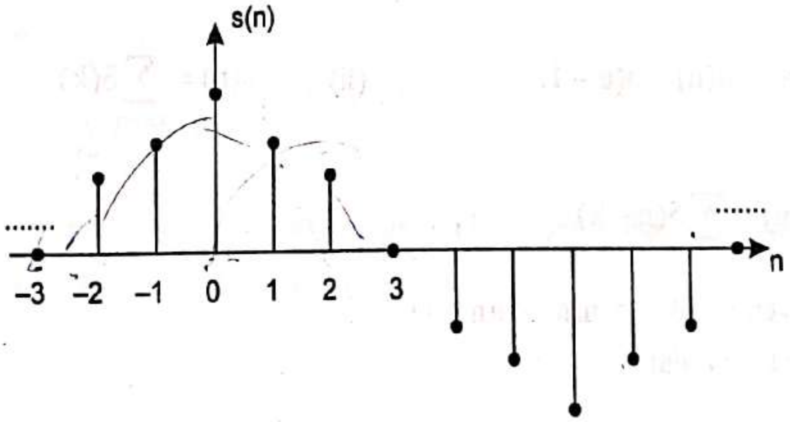


Fig. 2.7 *Graphical representation of cosine type sinusoidal sequence.*

2.4 Relationship Between Step, Ramp and Delta Functions

In this subsection, let us establish relationship between step, ramp and delta functions.

(i) Relation between unit step and unit ramp function : The relationship between unit step and unit ramp functions can be written as below

$$\frac{d}{dt}r(t) = u(t)$$

or
$$\boxed{\int u(t)dt = r(t)}$$

(ii) Relation between unit step and delta functions : The relationship between the unit step and delta functions can be written as below :

$$\frac{d}{dt}u(t) = \delta(t)$$

or
$$\int \delta(t)dt = u(t)$$

Hence, on integrating an unit impulse function, we get an unit step function.

(iii) Relation between unit ramp and delta functions : The relationship between unit ramp and delta functions can be written as below :-

$$r(t) = \int\int \delta(t)\, dt$$

or
$$\frac{d^2}{dt^2}r(t) = \delta(t)$$

Thus, on summarizing points (i), (ii) and (iii), we get

$$\delta(t) \xrightarrow{\text{Integrate}} u(t) \xrightarrow{\text{Differentiate}} r(t)$$

or
$$r(t) \xrightarrow{\text{Differentiate}} u(t) \xrightarrow{\text{Differentiate}} \delta(t)$$

Example 2.1 **Prove the following :**

(i) $\delta(n) = u(n) - u(n-1)$ **(ii)** $u(n) = \sum_{k=-\infty}^{n} \delta(k)$

(iii) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$

Solution : (i) Given : $\delta(n) = u(n) - u(n-1)$

We know that

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

so that
$$u(n-1) = \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n < 1 \end{cases}$$

Therefore, we have

$$u(n) - u(n-1) = \begin{cases} 0 & \text{for } n \geq 1 \text{ i.e., } n > 0 \\ 1 & \text{for } n = 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Note that the above equation is nothing but $\delta(n)$.

This means that

$$u(n) - u(n-1) = \delta(n)$$

$$= \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Hence Proved.

~~(ii)~~ Given
$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

We know that

$$\sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

Note that the right hand side of above equation is an unit sample sequence u(n).

Therefore, the given equation is proved.

~~(iii)~~ Given
$$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

We know that

$$\sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

Note that the right hand side of above equation is an unit sample sequences u(n).

Therefore, the given equation is proved.

2.5 Classification of Signals

Any investigation in signal processing is started with a classification of signals involved in the specific application. Signals can be classified in the following classes :

- Multichannel and Multidimensional signals
- Continuous-time and Discrete-time signals
- Analog and Digital signals
- Deterministic and Random signals
- Energy and Power signals
- Periodic and Non-periodic signals.
- Symmetric (even) and anti-symmetric (odd) signals.

2.5.1 Multichannel and Multidimensional Signals

Multichannel Signals. Signals which are generated by mutliple sources or multiple sensors are called Multichannel signals. These signals are represented by vector

$$s(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}.$$

Above signal represents a 3-channel signal. In electrocardiography, 3-lead and 12-lead electrocardiograph is often used in practice, which results in 3-channel and 12-channel signals, respectively.

Multidimensional Signal. A signal is called multidimensional signal if it is a function of M independent variables. *For example : Speech signal* is a one dimensional signal because amplitude of signal depends upon single independent variable, namely, time. *TV Picture Signal* : A B/W picture signal is an example of 2-dimensional signal because brightness of the signal at each point is a function of two spatial independent variable, namely, x and y. Variables x and y are width and height of the picture element.

A coloured picture signal is an example of 3-dimensional signal because brightness of the signal at each point is a function of three independent variables, namely, x, y and time (t).

2.5.2 Continuous-time and Discrete-time Signals

Continuous-time Signals. A signal that varies continuously with time is called *continuous-time signal.* These are defined for every value of independent variable, namely, time. For example *speech signal and temperature* of the room are continuous-time signals. Continuous-time signal is shown in Fig. 2.8.

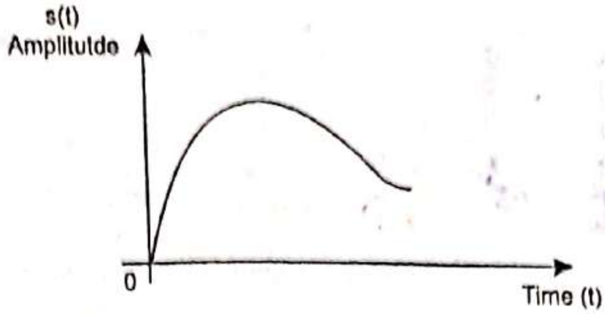


Fig. 2.8 *Continuous-time signal.*

Discrete-time Signal. Discrete-time signals are signals which are defined at discrete times (Fig. 2.9). These are represented by sequences of numbers. *For example :* Rail traffic signal is a discrete-time signal.

Discrete-time signals can be recovered by periodic sampling of continuous-time signals. Fig. 2.9 illustrates the discrete-time signal.

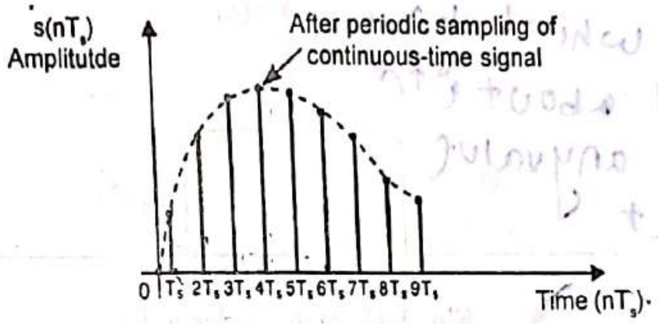


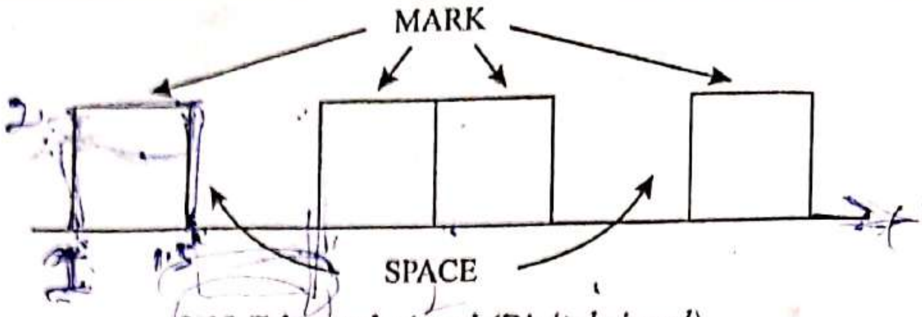
Fig. 2.9 *Discrete-time signal.*

2.5.3 Analog and Digital Signals

Analog Signals. Analog signals are signals whose both dependent variable and independent variable(s) are continuous in nature. Analog signals arise when a physical waveform is converted into an electrical signal. This conversion is performed by means of a transducer. *For example :* Telephone speech signals, TV signals etc., are very common types of analog signal.

Telephone Speech Signals. A telephone message comprises of speech sounds having vowels and consonants. These sounds produce an audio signal. These sound waves are converted into analog electrical signals by means of a transducer (microphone). Transducer is a device which converts non-electrical quantity into electrical signals. *Example :* Microphone. Continuous-amplitude, continuous-time signals are called *analog signals.*

Digital Signals. Digital signals are signals whose both dependent variable and independent variables are discrete in nature. Digital signals comprise of pulses occuring at discrete intervals of time. Telegraph and teleprinter signals are the example of digital signals. Fig.2.10 illustrates a telegraph signal.



2.10 *Telegraph signal (Digital signal).*

2.5.4 Deterministic and Random Signals

Deterministic Signals. A deterministic signal is one which -has no uncertainty with respect to its value at any value of independent variable, namely, time. *For Example :* Rectangular pulse given by Eqn (2.15) is a deterministic signal. Fig. 2.11 and Fig. 2.12 illustrate rectangular pulse and cosine signal respectively, both are the example of deterministic signal.

The signal which has certainity about its value w.r.t any value of independent variable.

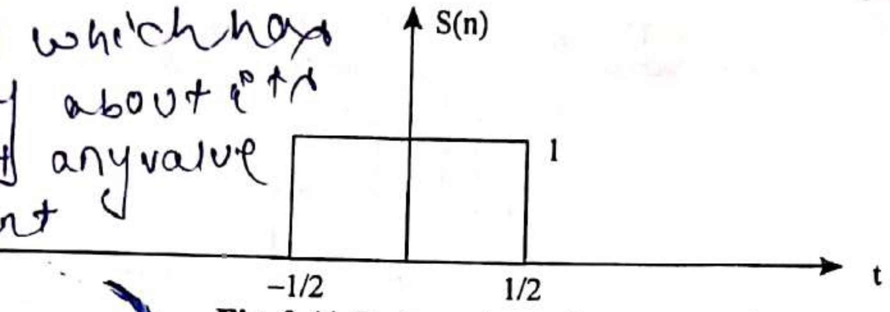


Fig. 2.11 *Rectangular pulse.*

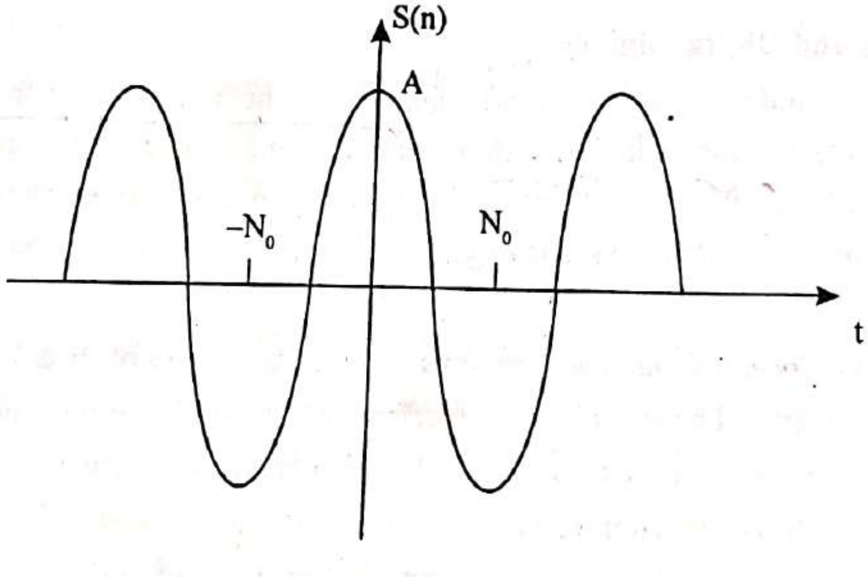


Fig. 2.12 *Cosine signal.*

$$s(n) = \begin{cases} 1, & |n| < \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases} \qquad ....(2.15)$$

Another example of deterministic Signal is sinusoidal signals such as sine waves and cosine waves as given in Eqn. (2.16)

$$s(n) = A\,Cos\omega n, \qquad -\infty < n < \infty \qquad ...(2.16)$$

Random Signal. A random signal is a signal which has some degree of uncertainty with respect to its value at any value of independent variable namely, time. *For example* : Thermal agitation noise in conductors is a random signal.

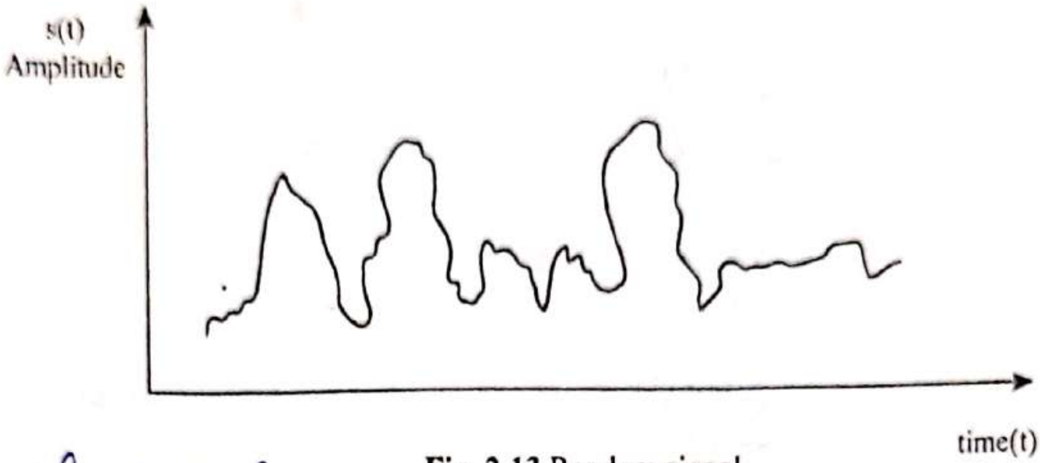


Fig. 2.13 Random signal.

classification of Discrete -Time signals :-

2.5.5 Energy signals and power signals

For a discrete-time signal x(n) the energy E is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(2.17)$$

The average power of a discrete-time signal x(n) is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \qquad ...(2.18)$$

The energy signal is one which has finite energy and zero average power.

Hence, x(t) is an energy signal, if :

$$\boxed{0 < E < \infty \text{ and } P = 0}$$

where, E is the energy and P is the power of the signal x(t).

The power signal, is one which has finite average power and infinite energy.

Hence, x(t) is a power signal, if:

$$\boxed{0 < P < \infty \text{ and } E = \infty}$$

However, if the signal does not satisfy any of the above two conditions, then it is neither an energy signal nor a power signal.

Example 2.2

Determine the values of power and energy of the following Signals. Find whether the signals are power, energy or neither energy nor power signals.

(i) $x(n) = \left(\frac{1}{3}\right)^n u(n)$ (ii) $x(n) = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}$ (iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$ (iv) $x(n) = e^{2n}u(n)$

Solution :

(i) Given $x(n) = \left(\frac{1}{3}\right)^n u(n)$

The energy of the signal

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty}\left[\left(\frac{1}{3}\right)^n\right]^2 \qquad \because u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n<0$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \qquad \boxed{1 + a + a^2 + \ldots\infty = \frac{1}{1-a}}$$

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

According to Geometric series,

$1 + x + x^2 + \cdots x^n$

$= \frac{1-x^{n+1}}{1-x}$

The power $P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=0}^{N}\left(\frac{1}{9}\right)^n$

$$\lim_{N\to\infty} \frac{1}{2N+1}\left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\frac{1}{9}}\right]$$

$= 0$

The energy is finite and power is zero. Therefore, the signal is energy signal.

(ii) $x(n) = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}$

$$E = \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2 \qquad \because \left| e^{j(\omega+\theta)} \right| = 1$$

$$E = \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)} \right|^2$$

$$= \sum_{n=-N}^{N} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$\lim_{N\to\infty} \frac{1}{2N+1}(2N+1) = 1$$

$$\sum_{n=-N}^{N} 1 = 2N+1$$

The energy is infinite and power is finite. Therefore, the signal is power signal.

(iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$

$$E = \sum_{n=-\infty}^{\infty} \left| \sin^2\left(\frac{\pi}{4}n\right) \right| = \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos\left(\frac{\pi}{2}n\right)}{2} \right] = \infty$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{2} = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| \sin^2\left(\frac{\pi}{4}n\right) \right|$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos\frac{\pi}{2}n}{2} = \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$(2N+1)$

$$= \frac{1}{2}$$

$$\boxed{\sum_{n=-N}^{N} 1 = 2N+1}$$

The energy is infinite and the power is finite. Therefore, the signal is a power signal.

(iv) $x(n) = e^{2n}u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} e^{4n} = 1 + e^4 + e^8 + ... + \infty = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{e^{4(N+1)-1}}{e^4 - 1}\right]$$

$$= \infty$$

The signal is neither power nor energy signal.

2.5.6 Periodic Signals and Aperiodic Signals

A signal is periodic with period N if and only if

$$x(n+N) = x(n) \text{ for all n.} \qquad ...(2.19)$$

The smallest value of N for which Eqn.(2.19) holds is known as fundamental period. If Eqn.(2.19) does not satisfy even for one value of n then the discrete-time signal is aperiodic.

A discrete-time sinusoidal signal is given by

$$x(n) = A \sin(\omega_0 n + \theta) \qquad ...(2.20)$$

The units of ω_0 and θ are radians.

The signal x(n) is periodic if and only if

$$x(n) = x(n+N) \text{ for all n.}$$

From Eqn.(2.20) we can obtain

$$x(n+N) = A \sin[\omega_0 (n+N) + \theta]$$

$$= A \sin[\omega_0 n + \omega_0 N + \theta] \qquad ...(2.21)$$

Eq. (2.20) and Eq. (2.21) are equal if

That is, there must an integer m such that

$\omega_0 N = 2\pi m$ or

$$\boxed{\omega_0 = 2\pi\left[\frac{m}{N}\right]} \qquad ...(2.22)$$

Therefore, the discrete time signal is periodic if the fundamental frequency ω_0 is rational multiple of 2π otherwise the discrete-time signal is aperiodic.

The sum of two periodic signals $x_1(n)$ and $x_2(n)$ with period N_1 and N_2 may or may not be periodic depending on the relationship between N_1 and N_2. If the sum to be periodic, the ratio of time periods $\boxed{\frac{N_1}{N_2}}$ must be a rational number or ratio of two integers. Otherwise the sum is not periodic.

Example 2.3

Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(i) $x(n) = e^{j6\pi n}$ (ii) $x(n) = e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$ (iii) $x(n) = \cos\frac{2\pi}{3}n$

(iv) $x(n) = \cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$

Solution

(i) $x(n) = e^{j6\pi n}$

$\omega_0 = 6\pi$. The fundamental frequency is multiple of π. Therefore, the signal is periodic.

From Eq. (2.22)

$$N = 2\pi\left[\frac{m}{\omega_0}\right]$$

$$= 2\pi\left[\frac{m}{6\pi}\right]$$

The minimum value of m for which N is integer is 3.

$$N = 2\pi\left[\frac{3}{6\pi}\right] = 1$$

Therefore, the fundamental period = 1.

(ii) $x(n) = e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$

$\omega_0 = \frac{3}{5}$, which is not a multiple of π. Therefore, the signal is aperiodic.

(iii) $x(n) = \cos\left(\frac{2\pi}{3}\right)n$

$$\omega_0 = \frac{2\pi}{3}$$

The signal is periodic.

The fundamental period

$$N = 2\pi\left[\frac{m}{\frac{2\pi}{3}}\right] = 3m$$

for $m = 1$

$N = 3$

Therefore, the fundamental period of the signal is 3.

(iv) $x(n) = \cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{3\pi}{4}\right)n$ $= x_1(n) + x_2(n)$

The fundamental period of the signal $\cos\left(\frac{3\pi}{4}\right)n$

$$N_1 = 2\pi\left[\frac{m}{\pi/3}\right] = 6 \text{ (for m =1)}$$

Similarly,

$$N_2 = 2\pi\left[\frac{m}{\frac{3\pi}{4}}\right] = 8 \qquad \text{(for m = 3)}$$

$$\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$$

$N = \text{LCM}(N_1, N_2)$
$= 24$

$$\Rightarrow \quad N = 4N_1 = 3N_2 = 24$$

$$N = 24.$$

2.5.7 Symmetric (even) and Antisymmetric (odd) signals

A discrete-time signal x(n) is said to be symmetric (even) signal if it satisfies the condition.

$$x(-n) = x(n) \text{ for all n.} \qquad ....(2.23)$$

Example : $x(n) = \cos \omega n$

The signal is said to be an odd signal if it satisfies the condition.

$$x(-n) = -x(n) \text{ for all n.} \qquad ....(2.24)$$

Example : A sin ωn

If x(n) is odd then x(0) = 0

A signal x(n) can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_0(n) \quad ....(2.25)$$

where $x_e(n)$ is even component of the signal and $x_0(n)$ is odd component of the signal.

Replace n by −n in Eq.(2.25) we get

$$x(-n) = x_e(-n) + x_0(-n) = xe(n) - x_0(n) \quad ...(2.26)$$

Adding Eq.(2.25) and Eq. (2.26) yields

$$2x_e(n) = x(n) + x(-n)$$

$$\boxed{\Rightarrow x_e(n) = \frac{1}{2}[x(n) + x(-n)]} \quad ...(2.27)$$

Similarly, we can get

$$\boxed{x_0(n) = \frac{1}{2}[x(n) - x(-n)]} \quad ...(2.27a)$$

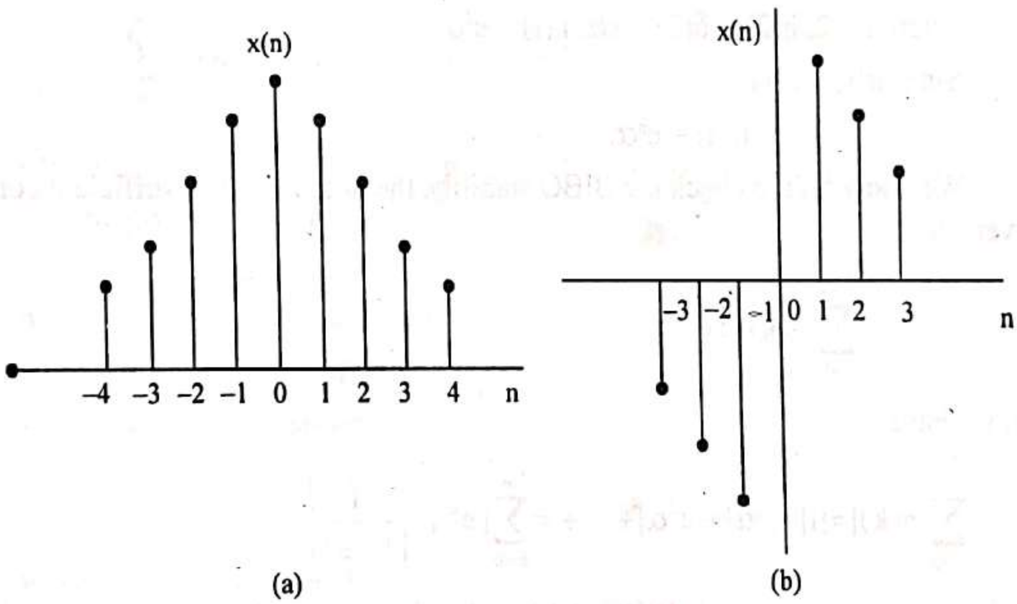


Fig 2.14 (a) A symmetric Signal (b) An antisymmetric Signal

A signal x(n) is said to be Causal if its value is zero for n < 0. Otherwise the signal is non-causal.

Examples for causal signals :

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$$

$$x_1(n) = a^n u(-n+1)$$

$$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$$

Example 2.4

A discrete-time system is characterized the following difference equation :

$$y(n) - x(n) + e^{\alpha} y(n-1)$$

Check this system for BIBO stability.

Solution : The given expression is

$$y(n) = x(n) + e\alpha\, y(n-1)$$

If $x(n) = S(n)$, then $y(n) = h(n)$.

Thus, the impulse response of the system will be

$$h(n) = 8(n) + e\alpha.\, h(n-1)$$

Now,

when $n = 0$, $h(0) = \delta(0) + e\alpha.\, h(-1) = 1$

when $n = 1$, $h(1) = \delta(1) + e\alpha.\, h(0) = e\alpha$

when $n = 2$, $h(2) = \delta(2) + e\alpha.\, h(1) = e^2\alpha$

Similarly, we have

$$h(n) = e^n \alpha.$$

We know that to check the BIBO stability, the necessary and sufficient condition is given by

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, we have

$$\sum_{k=0}^{\infty} |h(k)| = |1| + |e\alpha| + |e^2\alpha| + ... + = \sum_{k=0}^{\infty} |e^{ka}| = \left|\frac{1}{1-e^{\alpha}}\right|$$

Therefore, the given system is BIBO stable only when $e\alpha >< 1$ or $\alpha < 0$. (Ans)

Example 2.5

Check whether the following systems are BIBO stable or not :

(i) $y(n) = ax^2(n)$

(ii) $y(n) = ax(n) + b$

(iii) $y(n) = e^{-x(n)}$

Solution : (i) The given expression is

$$y(n) = ax^2(n)$$

If $x(n) = \delta(n)$

then $y(n) = h(n)$.

Thus, the impulse response is given by

$$h(n) = a\delta^2(n)$$

Now,

when $n = 0$, $h(0) = a\,\delta^2(0) = a$

when $n = 1$, $h(1) = a\,\delta^2(1) = 0$

In general, we have

$$h(n) = \begin{cases} a \text{ when } n = 0 \\ 0 \text{ when } n \neq 0 \end{cases}$$

We know that the necessary and sufficient condition for BIBO stability is expressed as

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

Here, we have

$$\sum_{k=0}^{\infty} |h(k)| = |h(0)| + |h(1)| + |h(2)| + \ldots\ldots + |h(k)| + \ldots. = |a|$$

Therefore, we conclude that the given system is BIBO stable only if $a < \infty$

(ii) The given system is

$$y(n) = a\,x(n) + b$$

If $x(n) = \delta(n)$ then

$$y(n) = h(n)$$

Thus, the impulse response is

$$h(n) = a\,\delta(n) + b$$

Now,

when $n = 0$, $h(0) = a\,\delta(0) + b = a + b$

when $n = 1$, $h(1) = a\,\delta(1) + b = b$

Here, $h(1) = h(2) = \ldots\ldots = h(k) = b$

Therefore, we have

$$h(n) = \begin{cases} a + b \text{ when } n = 0 \\ b \quad \text{when } n \neq 0 \end{cases}$$

Also, we know that the necessary and sufficient condition for BIBO stability is expressed as

$$\sum_{k=0}^{\infty}|h(k)| < \infty$$

Therefore, $$\sum_{k=0}^{\infty}|h(k)| = |h(0)| + |h(1)| + |h(2)| + \ldots\ldots + |h(k)| + \ldots.$$

or $$\sum_{k=0}^{\infty}|h(k)| = |a+b| + |b| + |b| + \ldots\ldots + |b| + \ldots$$

From above expression, it is obvious that this series never converges since the ratio between the successive terms is one.

Therefore the given system is **BIBO unstable.**

(iii) The given system is

$$y(n) = e^{-x(n)}$$

If $$x(n) = \delta(n)$$

then $$y(n) = h(n)$$

Thus, the impulse response is

$$h(n) = e^{-\delta(n)}$$

Now,

when $$n = 0, h(0) = e^{-\delta(0)} = e^{-1}$$

when $$n = 1, h(1) = e^{-\delta(1)} = e^{0} = 1$$

In general, we have

$$h(n) = \begin{cases} e^{-1} & \text{when } n = 0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

We know that the necessary and sufficient condition for BIBO stability is expressed as

$$\sum_{k=0}^{\infty}|h(k)| < \infty$$

Therefore, we have

$$\sum_{k=0}^{\infty}|h(k)| = |h(0)| + |h(1)| + |h(2)| + \ldots\ldots + |h(k)| + \ldots.$$

$$= e^{-1} + 1 + 1 + 1 + \ldots. + 1 \ldots.$$

From above equation, it is clear that the given system never converges, therefore, it is a BIBO unstable system.

Example 2.6

Check the BIBO stability for the impulse response of a discrete-time system given by

$$h(n) = a^n.u(n)$$

Solution : Given that $h(n) = a^n.\ u(n)$

This mean that $h(k) = a^k.\ u(k)$

We have $$\sum_{k=0}^{\infty} |h(k)| = |a^k| = |a^0| + |a^1| + |a^2| + \ldots\ldots + |a^k| + \ldots = \left|\frac{1}{1-a}\right|$$

From above, it is obvious that the given system is stable if a < 1, i.e., a lies inside the unit circle of the complex plane. Ans.

Example 2.7

Verify whether the following systems are BIBO stable or not

(i) $$h(t) = \begin{cases} \frac{1}{RC} e^{-t/RC} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

(ii) $$h(t) = \begin{cases} \frac{1}{\sqrt{LC}} \sin\left(-\frac{t}{\sqrt{LC}}\right) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Solution: (i) Given that $h(t) = \frac{1}{RC} e^{-t/RC}$

This is a causal system because we observe that

$$h(f) = 0 \text{ for } t < 0. \text{ (Given)}$$

For stability let us evaluate,

$$\int_{-\infty}^{\infty} h(t)dt = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC} dt \text{ for } t \geq 0$$

$$\int_{-\infty}^{\infty} h(t)dt = \int_{0}^{\infty} \frac{1}{RC} e^{-t/RC} dt$$

$$\int_{-\infty}^{\infty} h(t)dt = \frac{1}{RC}\left(-\frac{1}{1/RC}\right)\left[e^{-t/RC}\right]_0^{\infty} = 1 < \infty$$

Hence this system is stable.

(ii) $h(t) = \frac{1}{\sqrt{LC}} \sin\left(-\frac{1}{\sqrt{LC}}\right)$

This is causal system since

$h(t) = 0 \quad \text{for } t < 0 \text{ (Given)}$

For stability let us evaluate,

$$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{LC}} \sin\left(-\frac{1}{\sqrt{LC}}\right) dt \quad \text{for } t \geq 0$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{LC}} \sin\left(-\frac{1}{\sqrt{LC}}\right) dt$$

Let $\frac{1}{\sqrt{LC}} = p$. therefore $dt = \sqrt{LC}\ dp$.

Thus above equation becomes,

$$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{LC}} \sin(-p)\sqrt{LC}\,dp$$

$$\int_{0}^{\infty} \sin(p) = -[-\cos p]_0^{\infty} = \cos(\infty) - 1$$

The value of cosine function is always from – 1 to 1.

Here, since $\int_{-\infty}^{\infty} h(t)\,dt < \infty$, therefore, this is a stable system. (Ans)

2.6 Operation on Signals

Signal processing is a group of basic operations applied to an input signal resulting in another signal as the output. The mathematical transformation from one signal to another is represented as

$$y(n) = T[x(n)] \qquad ...(2.28)$$

The basic set of operations are

1. Shifting 2. Time reversal 3. Time scaling 4. Scalar multiplication 5. Signal multiplier 6. Signal addition

2.6.1 Shifting

The shift operation takes the input sequence and shift the values by an & integer increment of the independent variable. The shifting may delays or W advances the sequence in time. Mathematically this can be represented as

$$y(n) = x(n-k) \qquad \text{... (2.29)}$$

where x(n) is the input and y(n) is the output

If k is positive the shifting delays the sequence. If k is negative the shifting advances the sequence.

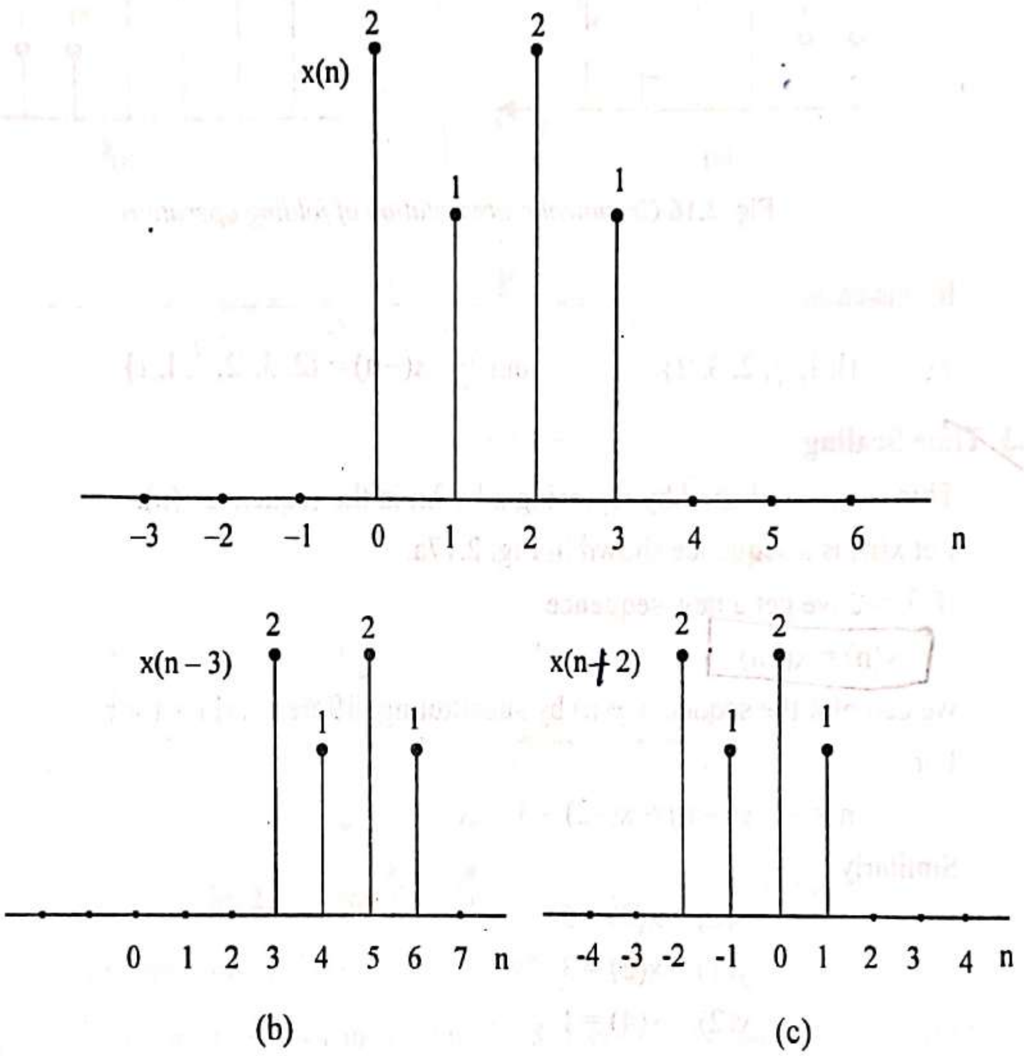


Fig. 2.15 Shift operation on signal

(a) Discrete time signal (b) delayed version (c) advanced version

A signal x(n) is shown in Fig. 2.15a. The signal x(n – 3) is obtained by shifting x(n) right by 3 units of time. The result is shown in Fig. 2.15b. On the other hand, the signal x(n + 2) is obtained by shifting x(n) left by two units of time (see Fig. 2.15c).

2.6.2 Folding or Time Reversal

This operation is an another useful scheme to develop a new sequence. In this operation independent variable n is replaced by –n. For example

$$y(n) = FD\,[x(n)] = x(-n) \quad \text{....(2.29)}$$

The figure 2.16 shows a graphical representation of folding operation.

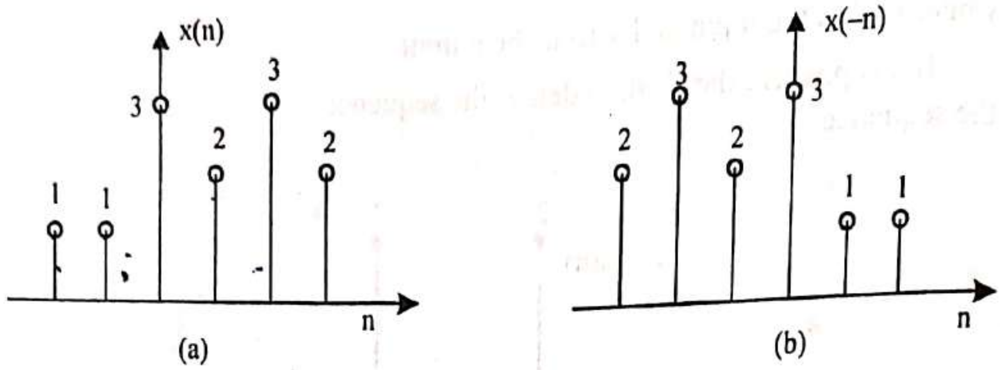


Fig. 2.16 *Graphical representation of folding operation*

In this case,

$x(n) = \{1, 1, \underset{\uparrow}{3}, 2, 3, 2\}$ and $x(-n) = \{2, 3, 2, \underset{\uparrow}{3}, 1, 1\}$

2.6.3 Time Scaling

This is accomplished by replacing n by λn in the sequence x(n).

Let x(n) is a sequence shown in Fig. 2.17a.

If $\lambda = 2$ we get a new sequence

$$\boxed{y(n) = x(2n)}$$

we can plot the sequence y(n) by substituting different values for n.

For

$$n = -1;\ y(-n) = x(-2) = 3$$

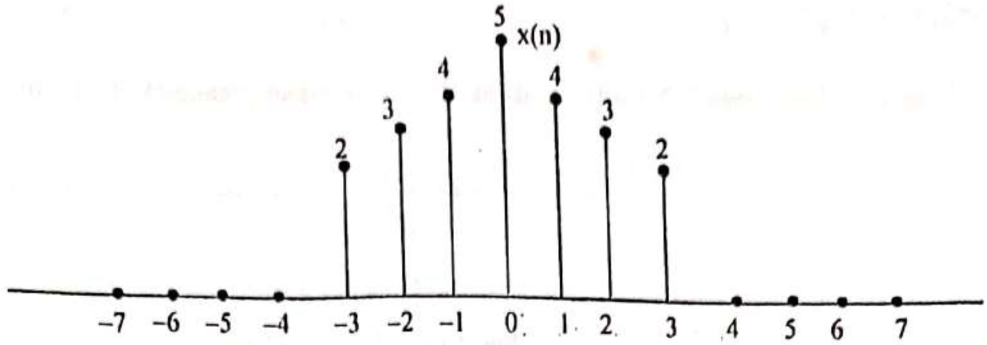
Similarly

$$y(0) = x(0) = 5$$
$$y(1) = x(2) = 3$$
$$y(2) = x(4) = 1$$

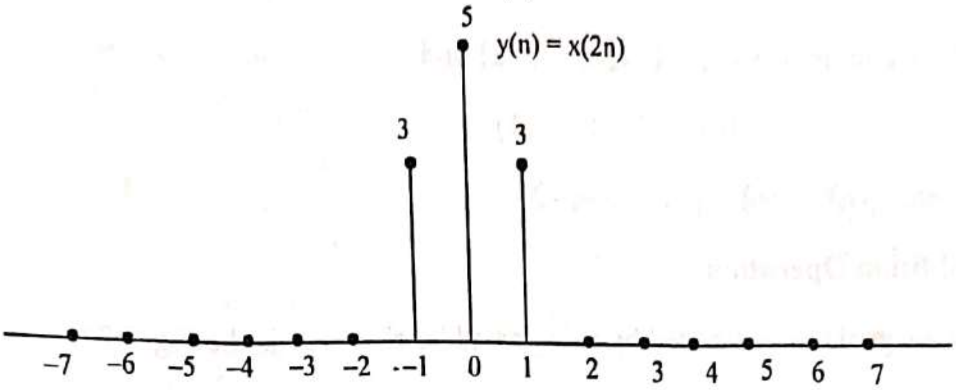
so on.

From the above result we can conclude that, to plot y(n) we have to skip the odd-numbered samples in x(n) and retain even-numbered samples. The resulting sequence is shown in Fig. 2.17b.

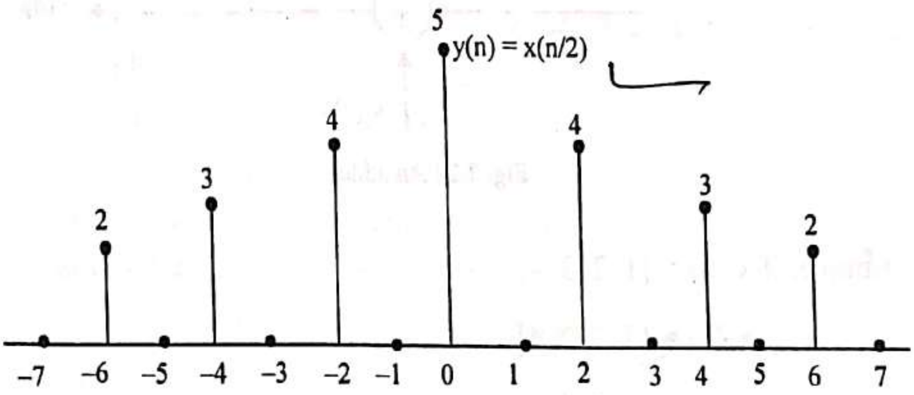
The original sequence x(n) is obtained by sampling a continuous signal x(t). The signal x(2n) is obtained by reducing the sampling rate on the continuous-time signal by a factor of 2. This process of reducing sampling rate is often referred as down sampling or decimation.



(a)



(b)



(c)

Fig. 2.17 *Graphical Representation of time scaling*

2.6.4 Scalar Multiplication or Amplitude Scaling

A scalar multiplier is shown in the Fig. 2.18. Here the signal x(n) is multiplied by a scalefactor A.

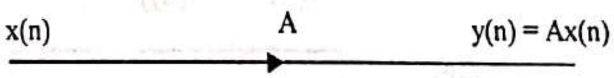


Fig. 2.18 A scale Multiplier

For example if $x(n) = \{1, 2, 1, -1\}$ and $A = 3$

Then the signal $Ax(n) = \{3, 6, 3, -3\}$

2.6.5 Signal Multiplier

Fig. 2.19 illustrates the multiplication of two signal sequences to form another sequence.

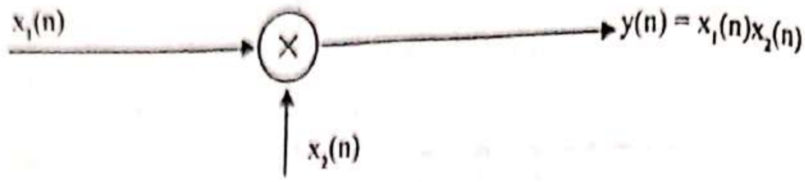


Fig. 2.19 A signal multiplier

For example, if $x_1(n) = \{-1, 2, -3, -2\}$ and

$$x_2(n) = \{1, -1, -2, 1\}$$

Then, $x_1(n) \cdot x_2(n) = \{-1, -2, 6, -2\}$

2.6.6 Addition Operation

Two signals can be added by using an adder shown as in the Fig. 2.20.

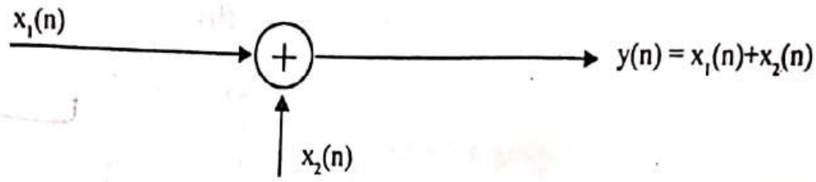


Fig. 2.20 An adder

For example, if $x_1(n) = \{1, 2, 3, 4\}$

$$x_2(n) = \{4, 3, 2, 4\}$$

Then, $x_1(n) + x_2(n) = \{5, 5, 5, 8\}$

2.7 Discrete-Time System

A discrete-time system is a device or algorithm that operates on a discrete-time input signal x(n), according to some well-defined rule, to produce another discrete-time signal y(n) called the output signal. The relationship between x(n) and y(n) is

$$y(n) = T[x(n)]$$

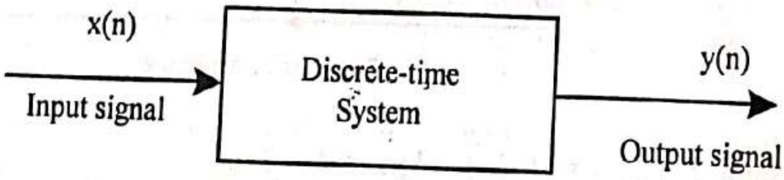


Fig. 2.21 Discrete time system

2.8 Classification of discrete-time systems

Discrete-time systems are classified according to their general properties and characteristics. They are

(1) Static and dynamic systems,
(2) Time-variant and time-invariant systems.
(3) Causal and non-causal systems,
(4) Linear and non-linear systems.
(5) FIR and IIR systems.
(6) Stable and unstable systems.

2.8.1 Static and Dynamic Systems

A discrete-time system is called static or memory less if its output at any instant n depends on the input samples at the same time, but not on past or future samples of the input. In any other case, the system is said to be dynamic or to have memory.

The systems described by the following equations

$$y(n) = ax(n)$$
$$y(n) = ax^2(n)$$

are both static as they won't require memory. On the other hand, the systems described by the following equations

$$y(n) = x(n-1) + x(n-2)$$
$$y(n) = x(n) + x(n-1)$$

are dynamic systems as they require finite memory.

2.8.2 Time-Variant and Time-invariant systems

A, system is called time-invariant if its input-output characteristics do not change with time.

To test if any given system is time-invariant, first apply an arbitrary sequence x(n) and find y(n). Now delay the input sequence by k samples and find output sequence, denote it as

$$y(n, k) = T[x(n-k)]$$

Delay the output sequence by k samples, denote it as $y(n-k)$. If

$$\boxed{y(n, k) = y(n-k)} \text{ Time invariant}$$

for all possible values of k, the system is time-invariant on the otherhand, if the output.

$$\boxed{y(n, k) \neq y(n-k)} \text{ Time variant}$$

even for one value of k, the system is time-variant.

Example 2.8

Determine if the following systems are time-invariant or time-variant.

(i) $y(n) = x(n) \sin \omega_0 n$ (ii) $y(n) = x(-n)$

Solution :

(i) Given $y(n, k) = x(n-k) \sin \omega_0 n$ $y(n,k) = T[x(n-k)]$

If we delay the output by K unit in time then

$$y(n-k) = x(n-k) \sin \omega_0 (n-k)$$

Since $y(n, k) \neq y(n-k)$ the system is time variant.

(ii) If the input is delayed by k units in time and applied to the system we have

$$y(n, k) = T[x(n-k)] = x(-n-k)$$

If the output is delayed by k samples

$$y(n-k) = x[-(n-k)] = x(-n+k)$$

Here

$$y(n,k) \neq y(n-k)$$

so, the system is time-variant.

2.8.3 Causal and Non-Causal Systems

causal system = Recursive system.

A system is said to be causal if the output of the system at any time n depends only on present and past inputs, but does not depend on future inputs. This can be represented mathematically as

$$y(n) = F[x(n), x(n-1)\ x(n-2)\ ]$$

If a system depends not only on present and past inputs but also on future inputs then it is said to be a non-causal system.

Example 2.9

Determine if the system described by the following equations are

causal or non-causal. (i) $y(n) = x(n) + \frac{1}{x(n-1)}$ (ii) $y(n) = x(n^2)$

Solution

(i) Given $y(n) = x(n) + \frac{1}{x(n-1)}$

For n = – 1

$$y(-1) = x(-1) + \frac{1}{x(-2)}$$

For n = 0

$$y(0) = x(0) + \frac{1}{x(-1)}$$

For n = 1

$$y(1) = x(1) + \frac{1}{x(0)} \quad + x(2)$$

For all the values of n the output depends on present and past inputs. Therefore, the system is causal.

(ii) $y(n)=x(n^2)$

For n = –1

$$y(-1) = x(1)$$

For n = 0

$$y(0) = x(0)$$

For n = 1

$$y(1) = x(1)$$

For negative values of n, the system depends on future inputs. So, the system is non-causal.

2.8.4 Linear and Non-Linear Systems

A system that satisfies the superposition principle is said to be a linear system, superposition principle states that the response of the system to a weighted of signals be equal to the corresponding weighted sum of the outputs of system to each of the individual input signals.

A system is linear if and only if

$$T[a_1x_1(n) + a_2x_2(n)\ a_1x_2(n)] + a_2T[x_2(n)]$$

$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$

for any arbitrary constants a_2 and a_2.

A relaxed system that does not satisfy the superposition principle is called non-linear.

Example 2.10

Determine if the system described by the following input-output equations are linear or non-linear.

(i) $y(n) = x(n)+ \dfrac{1}{x(n-1)}$ (ii) $y(n) = x^2(n)$ (iii) $y(n) = x(n) + u(n+1)$

Solution :

Rn L.H.S. side

(i) Given $y(n) = x(n) + \dfrac{1}{x(n-1)}$

x(n)

For two input sequences $x_1(n)$ and $x_2(n)$ the corresponding outputs are

$x_1(n)$ $x_2(n)$

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)}$$

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)}$$

The output due to weighted sum 01 inputs is

$$y_3(n) = T[a_1\ x_1(n) + a_2x_2(n)]$$

$$= a_1x_1(n) + a_2x_2(n) + \frac{1}{a_1x_1(n-1) + a_2x_2(n-1)} \qquad ...(2.30)$$

on the other hand, the linear combination of the two outputs is

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n) + \frac{a_1}{x_1(n-1)} + a_2x_2(n) + \frac{a_2}{x_2(n-1)} \qquad ...(2.31)$$

Eq. (2.30) and Eq. (2.31) are not equal, superposition principle is not satisfied. So the system is non-linear,

(ii) $y(n) = x^2(n)$

The outputs due to the signals $x_1(n)$ and $x_2(n)$ are

$$y_1(n) = T[x_1(n)] = x_1^2(n)$$

$$y_2(n) = T[x_2(n)] = x_2^2(n)$$

The weighted sum of outputs is

$$a_1T[x_1(n)] + a_2T[x_2(n)] = a_1x_1^2(n) + a_2x_2^2(n) \qquad ...(2.32)$$

The output due to weighted sum of inputs is

$$y_3(n) = T\ [a_1x_1(n) + a_2x_2(n)\ ] = [a_1x_1n) + a_2x_2(n)\ ] \qquad ...\ (2.33)$$

Eq. (2.32) and Eq. (2.33) are not equal, superposition principle is not satisfied. So, the system is non-linear.

(iii) $y(n) = x(n) + u(n+1)$

Consider

$$y_1(n) = x_1(n) + u(n+1)$$

$$y_2(n) = x_2(n) + u(n+1)$$

linear combination of the two input sequences results in the output

$$y_3(n) = T\ [a\ x_1(n) + b\ x_2(n)]$$

$$= [a\ x_1(n) + b\ x_2(n)] + u(n+1) \qquad ....(2.34)$$

Finally the linear combination of two outputs yields

$$ay_1(n) + by_2(n) = ax_1(n) + b\ y(n+1) + bx_2\ (n) + b\ u(n+1) \qquad ....(2.35)$$

Since Eqn. (2.34) and (2.35) is not same, so the system is nonlinear.

2.8.5 FIR and IIR Systems

Linear time-invariant systems can be classified according to the type of impulse response. If the impulse response sequence is of finite duration, the system is called a finite impulse-response (FIR) system. On the otherhand, an infinite impulse response (IIR) system has an impulse response that is of infinite duration.

An example of a FIR system is described by

Lineart time invariant

$$h(n) = \begin{cases} -1 & n = 1,2 \\ 1 & n = 14 \\ 0 & \text{otherwise} \end{cases}$$

An example of an IIR system is described by

$$h(n) = nu(n)$$

2.8.6 Stable and unstable systems

An LTI system is stable if it produces a bounded output sequence for every bounded input sequence. If, for some bounded input sequence x(n), the output is unbounded (infinite), the system is classified as unstable. Let x(n) be a bounded input sequence, h(n) be the impulse response of the system and y(n) be the output sequence. Taking the magnitude of the output

$$\text{we have } |y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)\,x(n-k)\right|$$

We know that the magnitude of the sum of terms is less than or equal to the sum of the magnitudes, hence

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\,.\,|x(n-k)|$$

Let the bounded value of the input is equal to M, the Eqn. can be written as

$$|y(n)| \le M \sum_{k=-\infty}^{\infty} h(k)$$

The above condition will be satisfied when

system is stable if it has BIBO.

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

So, the necessary and sufficient condition for stability is

$$\text{Stability condition } \boxed{\sum_{n=-\infty}^{\infty} |h(n)| < M < \infty}$$

Example 2.11

Test the stability of the system whose impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Solution :

The necessary and sufficient condition for stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

Given $h(n) = (1/2)^n u(n)$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |(1/2)^n u(n)|$$

$$= \sum_{n=0}^{\infty} (1/2)^n$$

$$= 1 + 1/2 + 1/2^2 \ldots \infty \qquad \boxed{\left(\because 1 + a + a^2 \ldots \infty = \frac{1}{1-a}\right)}$$

$$= \frac{1}{1 - 1/2} = 2 < \infty$$

Hence the system is stable.

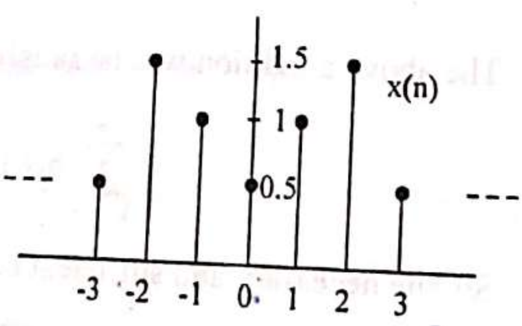
2.9 Representation of an Arbitrary Sequence

Any arbitrary sequence x(n) can be represented in terms of delayed and scaled impulse sequence $\delta(n)$. Let x(n) is an infinite sequence as shown in Fig. 2.22a.

The sample x(0) can be obtained by multiplying x(0), the magnitude, with unit impulse $\delta(n)$ as shown in Fig. 2.22c.

$$\text{i.e., } x(0)\delta(n) = \begin{cases} x(-1) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

Similarly, the sample $x(-1)$ can be obtained by multiplying $x(-1)$ the magnitude, with one sample advanced unit impulse $\delta(n+1)$ as shown in Fig. 2.22d.



(a)

i.e, $x(-1)\delta(n+1) = \begin{cases} x(-1) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$

In the same way

$$x(-2)\delta(n+2) = \begin{cases} x(-2) & \text{for } n = -2 \\ 0 & \text{for } n \neq -2 \end{cases}$$

$$x(1)\delta(n-1) = \begin{cases} x(1) & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$x(2)\delta(n-2) = \begin{cases} x(2) & \text{for } n = 2 \\ 0 & \text{for } n \neq 2 \end{cases}$$

The sum of the five sequences in the Fig. 2.22a

$x(-2)\delta(n+2) + x(-1)\delta(n+1)$

$+x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2)$

equal $x(n)$ for $-2 \leq n \leq 2$. In general we can write $x(n)$ for $-<\infty \leq n \leq \infty$ as

$$x(n) = \ldots x(-3)\delta(n+3) + x(-2)\delta(n+2)$$
$$+x(-1)\delta(n+1) + x(0)\delta(n)$$
$$+ x(3)\delta(n-3) + \ldots$$

$$\boxed{= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)} \qquad \ldots (2.36)$$

where $\delta(n-k)$ is unity for $n = k$ and zero for all other terms.

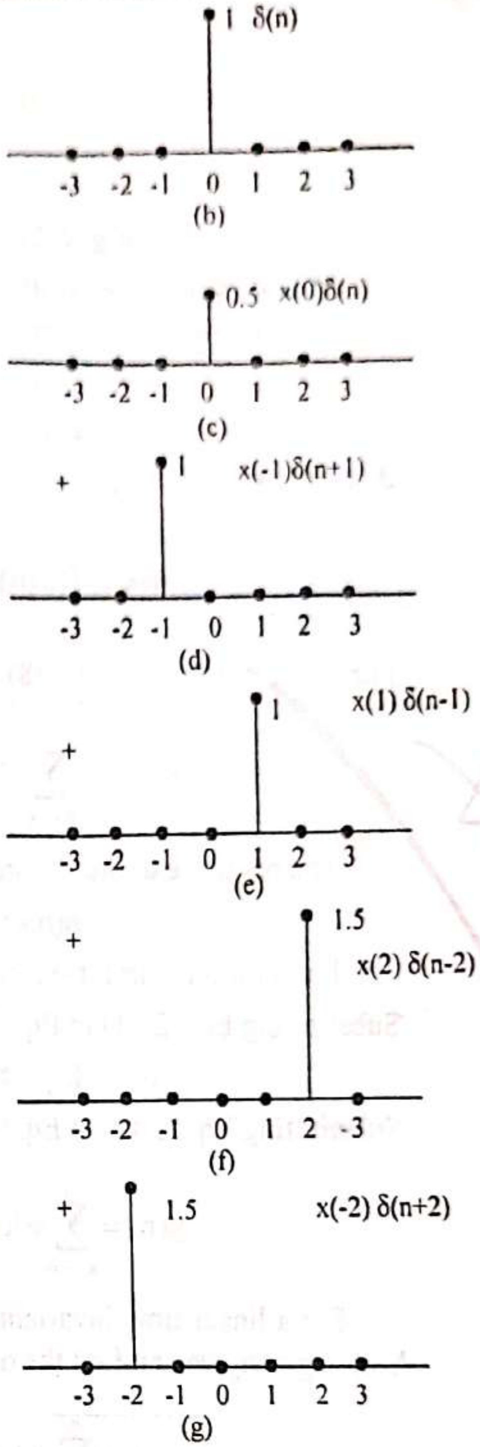


Fig. 2.22. *Representationof a sequence as a sum of delayed impulses*

2.10 Impulse Response and Convolution Sum

A discrete-time system performs an operation on an input signal based on a pre-defined criteria to produce a modified output signal. The input signal $x(n)$ is the system excitation, and $y(n)$ is the system response. This transform operation is shown in Fig. 2.23.

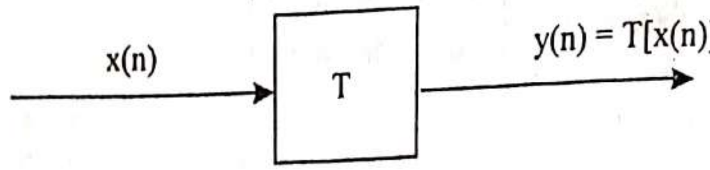


Fig. 2.23. *A Discrete - time system representation*

If the input to the system is a unit impulse i.e., $x(n) = \delta(n)$ then the output of the system is known as impulse response denoted by *h(n)* where

$$(n) = T[\delta(n)] \qquad \text{... (2.37)}$$

We know that any arbitrary sequence x(n) can be represented as a weighted sum of discrete impulses (Eq. 2.36). Now the system response is given by

$$y(n) = T[x(n)] = T\,[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)] \qquad \text{... (2.38)}$$

For a linear system Eq. (2.38) reduces to

$$y(n) = [\sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)] \qquad \text{....(2.39)}$$

The response to the shifted impulse sequence can be denoted by h(n, k) is defined as

$$h(n,k) = T)[\delta(n-k)] \qquad \text{...(2.40)}$$

For a time-invariant system $h(n, k) = h(n - k)$ ...(2.41)

Substituting Eq. (2.41) in Eq. (2.40) we obtain

$$T[\delta(n-k)] = h(n-k) \qquad \text{...(2.42)}$$

Substituting Eq. (2.42) in Eq. (2.39) we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad \text{...(2.43)}$$

For a linear time-invariant system if the input sequence x(n) and impulse response h(n) is given, we can find the output y(n) by using the equation

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)} \qquad \text{... (2.44)}$$

which is known as convolution sum and can be represented as

$y(n) = x(n) * h(n)$. where $*$ denotes the convolution operation.

The convolution sum of two sequences can be found by using following steps.

Step 1: Choose an initial value of n, the starting time for evaluating the output sequence y(n). If x(n) starts at $n = n_1$ and h(n) starts at $n = n_2$ then $n = n_1 + n_2$ is a good choice.

Step 2: Express both sequences in terms of the index k.

Step 3: Fold h(k) about k = 0 to obtain h(–k) and shift by n to the right if n is positive and left if n is negative to obtain h(n – k).

Step 4: Multiply the two sequences x(k) and h(n – k) element by element and sum the products to get y(n).

Step 5: Increment the index n, shift the sequence h(n – k) to right by one sample and do Step 4.

Step 6: Repeat Step 5 until the sum of products is zero for all remaining values of n.

Properties of Convolution

(i) Commutative Law: $x(n) * h(n) = h(n) * x(n)$

(ii) Associative Law: $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_1(n)]$

(iii) Distributive Law: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n)$ + $x(n) * h_2(n)$

Example 2.12

Determine the response of a dsicrete-time system to input signal $s(n) = \{\underset{\uparrow}{2}, 1, 3, 1\}$.

Also given unit-sample (impulse) response

$$h(n) = \{1, \underset{\uparrow}{2}, 2, -1\}$$

Solution. Convolution sum is defined as

$$y(n) = \sum_{k=-\infty}^{\infty} s(k)h(n-k)$$

n = 0, $$y(0) = \sum_{k=-\infty}^{\infty} s(k)h(-k)$$

$$s(k) = \underset{\uparrow}{2},\ 1,\ 3,\ 1$$

$$h(k) = 1,\ \underset{\uparrow}{2},\ 2,\ -1$$

$$\boxed{\begin{array}{ll} s(k) = & \underset{\uparrow}{2},\ 1,\ 3,\ 1 \\ h(-k) = & -1,\ 2,\ 2,\ 1 \end{array}}$$

$$y(0) = \sum_{k=-\infty}^{\infty} s(k)h(-k) = 2 \times 2 + 1 \times 1 = 4 + 1 = 5$$

2, 1, 3, 1
-1, 2, 2, 1
h(1-k)
2, 1, 3, 1
2 2 1 →

n = 1, $$y(1) = \sum_{k=-\infty}^{\infty} s(k)h(1-k)$$

s(k) =	2, 1, 3, 1
	↑
h(1–k) =	–1, 2, 2, 1

$$y(1) = \sum_{k=-\infty}^{\infty} s(k)h(1-k) = 2\times 2 + 1\times 2 + 3\times 1$$

$$= 4 + 2 + 3 = 9$$

$$n = 2, \quad y(2) = \sum_{k=-\infty}^{\infty} s(k)h(2-k)$$

s(k) =	2, 1, 3, 1
	↑
h(2–k) =	–1, 2, 2, 1

$$y(2) = \sum_{k=-\infty}^{\infty} s(k)h(2-k) = 2\times(-1) + 1\times 2 + 3\times 2$$

$$+ 1\times 1 = -2 + 2 + 6 + 1 = 7$$

$$n = 3, \quad y(3) = \sum_{k=-\infty}^{\infty} s(k)h(3-k)$$

s(k) =	2, 1, 3, 1
	↑
h(3–k) =	–1, 2, 2, 1

$$y(3) = \sum_{k=-\infty}^{\infty} s(k)h(3-k) = 1\times(-1) + 3\times 2 + 1\times 2$$

$$= -1 + 6 + 2 = 7$$

$$n = 4, \quad y(4) = \sum_{k=-\infty}^{\infty} s(k)h(4-k)$$

s(k) =	2, 1, 3, 1
	↑
h(4–k) =	–1, 2, 2, 1

$$y(4) = \sum_{k=-\infty}^{\infty} s(k)h(4-k) = 3\times(-1) + 1\times 2$$

$$n = 5, \quad y(5) = \sum_{k=-\infty}^{\infty} s(k)h(5-k)$$

$$\boxed{\begin{array}{lllllll} s(k) = & \underset{\uparrow}{2}, & 1, & 3, & 1 & & \\ h(5-k) = & & & -1, & 2, & 2, & 1 \end{array}}$$

$$y(5) = \sum_{k=-\infty}^{\infty} s(k)h(5-k) = 1 \times (-1) = -1$$

$n = 6, y(6) = 0$
$n = 7, y(7) = 0$

If sequences s(n) and h(n) have M sample points and N sample points, respectively, then convolution of these sequences will have $M + N - 1$ sample points. In this example sequence s(n) has 4 points, and sequence s(n) has 4 points.

Then convolution of these sequences will have $4 + 4 - 1 = 7$ points

$n = -1,$ $\qquad y(-1) = \sum_{k=-\infty}^{\infty} s(k)h(-1-k)$

$$\boxed{\begin{array}{lllllll} s(k) = & & & \underset{\uparrow}{2}, & 1, & 3, & 1 \\ h(-1-k) = & -1, & 2, & 2, & 1 & & \end{array}}$$

$$y(-1) = \sum_{k=-\infty}^{\infty} s(k)h(-1-k) = 2 \times 1 = 2$$

Resultant of convolution sum of s(n) and h(n) is y(n) and is given as follows :

$$y(n) = \{y(-1), y(0), y(1), y(2), y(3), y(4), y(5)\}$$
$$= \{2, \underset{\uparrow}{5}, 9, 7, 7, -1, -1\}$$

2.11 Properties of Convolution Sum

Convolution is a mathematical operation between two signal sequences s(n) and h(n). This operation satisfies following properties :

1. Commutative law
2. Associative law
3. Distributive law.

Commutative Law. Convolution sum satisfies commutative law. According to commutative law for a system shown in Fig. 2.24.

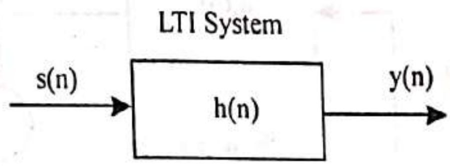


Fig. 2.24 LTI system

$$s(n) * h(n) = h(n) * s(n)$$

or $$\sum_{k=-\infty}^{\infty} s(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)s(n-k)$$

This is true only for LTI discrete-time systems.

Associative Law. Convolution sum also satisfies the associative law. According to associative law for the systems shown in Fig. 2.25.

$$[s(n) * h_1(n)] * h_2(n) = s(n) * [h_1(n) * h_2(n)]$$

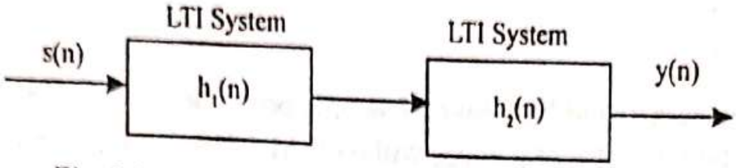


Fig. 2.25 *Cascading of two discrete-time LTI systems.*

Distributive Law. This law is also satisfied by convolution sum of two-discrete-time LTI systems. According to the distributive law for the systems shown in Fig. 2.26.

$$s(n) * [h_1(n) + h_2(n)] = s(n) * h_1(n) + s(n) * h_2(n)$$

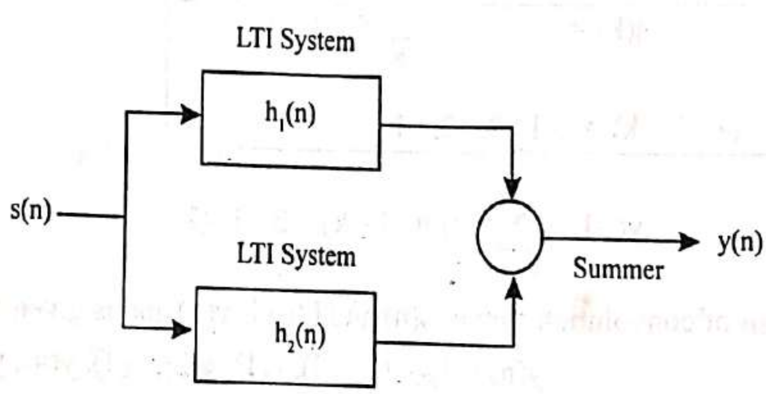


Fig. 2.26 *Two discrete-time LTI systems in parallel.*

2.12 Inter connection of LTI Systems

2.12.1 Parallel connection of systems

Consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in parallel as shown in Fig. 2.27(a).

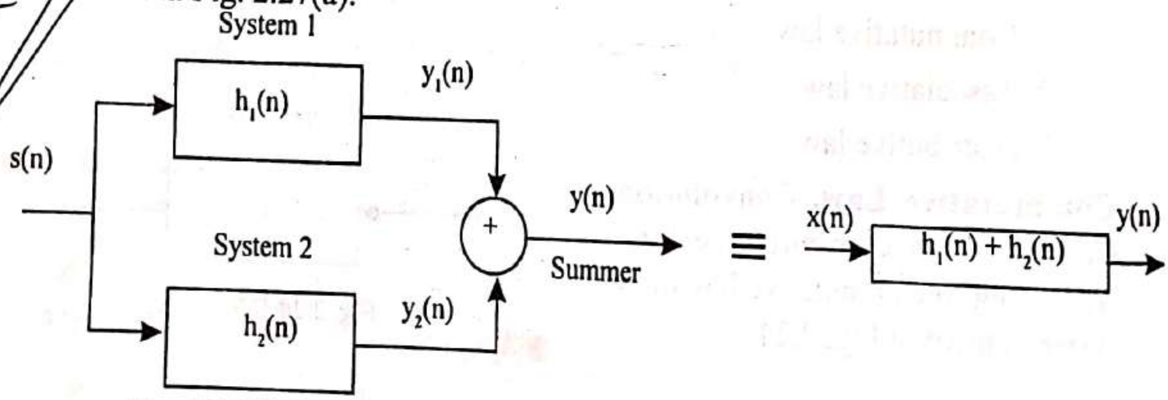


Fig. 2.27(a) *Parallel connection of two system; (b) Equivalent system*

From Fig. 2.27(a) the output of system 1 is

$$y_1(n) = x(n) * h_1(n) \qquad .....(2.45)$$

and the output of system 2 is

$$y_2(n) = x(n) * h_2(n) \qquad .....(2.46)$$

The output

$$y(n) = y_1(n) * y_2(n)$$

$$= x(n) * h_1(n) + x(n) * h_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h_1(n-k) = \sum_{k=-\infty}^{\infty} x(k)h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k) + h_2(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

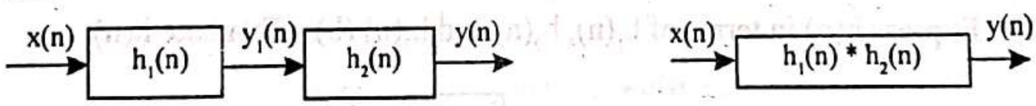
$$= x(n) * h(n)$$

where $h(n) = h_1(n) + h_2(n)$. .....(2.47)

Thus if the two-systems are connected in parallel the overall impulse response is equal to sum of two impulse responses.

2.12.2 Cascade Connection of Two Systems

Consider two LT1 systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in cascade. Let.



(a) (b)

Fig. 2.28 (a) Cascade connection of two systems; (b) Equivalent system

$y_1(n)$ is the output of the first system. Then

$$y_1(k) = x(k) * h_1(k)$$

$$= \sum_{v=-\infty}^{\infty} x(v)h_1(k-v) \qquad .....(2.48)$$

the output

$$y(n) = y_1(k) * h_2(k)$$

$$= \left[\sum_{v=-\infty}^{\infty} x(v)h_1(k-v)\right] * h_2(k) \qquad .....(2.49)$$

$$y(n) = \sum_{k=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} x(v)h_1(k-v)h_2(n-k)$$

Let $\quad k - v = p$

$$y(n) = \sum_{v=-\infty}^{\infty} x(v) \sum_{p=-\infty}^{\infty} h_1(p)h_2(n-v-p)$$

$$= \sum_{v=-\infty}^{\infty} x(v)h(n-v)$$

$$= x(n) * h(n)$$

$$\text{where } h(n) = \sum_{v=-\infty}^{\infty} h_1(k)h_2(n-k)$$

$$= h_1(n) * h_2(n) \qquad .....(2.50)$$

Hence the impulse Response of two LT1 systems connected in cascade is the convolution of the individual impulse responses.

Example 2.13

An inter connection of LT1 systems is shown in Fig. 2.29.

The impulse responses are $h_1(n) = \left(\frac{1}{2}\right)^n [\,u(n) - u(n-3)\,]$; $h_2(n) = \delta(n)$

and $h_3(n) = u(n - I)$. Let the impulse response of the overall system from x (n) to y(n) be denoted as h(n).

(a) Express h(n) in terms of $h_1(n)$, $h_2(n)$ and $h_3(n)$ (b) Evaluate h(n).

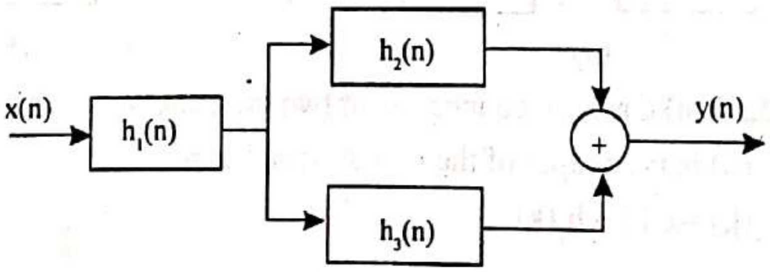


Fig. 2.29

Solution

The systems with impulse responses $h_2(n)$ and $h_3(n)$ are connected in parallel. This can be replaced system an equivalent system whose impulse response is sum of two individual impulse responses. That is

$$h'(n) = h_2(n) + h_3(n)$$

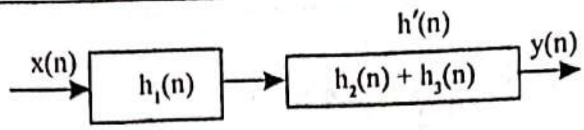


Fig. 2.30

Now the systems with impulse responses $h_1(n)$ and $h'(n)$ is connected in cascade. Therefore, the overall impulse response

$$\begin{aligned} h(n) &= h_1(n) * h'(n) \\ &= h_1(n) * [h_2(n) + h_3(n)] \\ &= h_1(n) * h_2(n) + h_1(n) * h_3(n) \end{aligned}$$

Given $h_1(n) = \left(\frac{1}{2}\right)^n \; [\, u(n) - u(n+3)\,]$

$h_2(n) = \delta(n)$

$h_3(n) = u(n-1)$

$h_1(n) * h_2(n)$

$$= \left[\left(\frac{1}{2}\right)^n [u(n) - u(n-3)]\right] * \delta(n)$$

$$= \left(\frac{1}{2}\right)^n [u(n) - u(n-3)] \quad \text{1} \qquad \boxed{\because x(n) * \delta(n) = x(n)}$$

$h_1(n) * h_3(n)$

$$= \left\{\left(\frac{1}{2}\right)^n [u(n) - u(n-3)]\right\} * u(n-1)$$

$$= \left(\frac{1}{2}\right)^n u(n) * u(n-1) - \left(\frac{1}{2}\right)^n u(n-3) * u(n-1)$$

Let us take

$$y_1(n) = \left(\frac{1}{2}\right)^n u(n) * u(n-1) \qquad 0 + 1 = 1$$

$$y_1(n) = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^n \text{ for } n \geq 1$$

$$= 0 \qquad \text{for } n < 1$$

$$\sum_{n=0}^{n} a^n = \frac{1-a^{n+1}}{1-a}$$

$$\Rightarrow y_1(n) = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$\Rightarrow y_1(n) = 2\left[1-\left(\frac{1}{2}\right)^n\right] \text{for } n \geq 1$$

$$= 0 \qquad \text{for } n < 1$$

Therefore,

$$y_1(n) = 2\left[1-\left(\frac{1}{2}\right)^n\right]u(n-1)$$

$$y_2 n = \left(\frac{1}{2}\right)^n u(n-3) * u(n-1)$$

$$= \sum_{k=3}^{n-1}\left(\frac{1}{2}\right)^n \text{ for } n \geq 4 \qquad \boxed{3+1=4}$$

$$= 0 \qquad \text{for } n < 4$$

$$\Rightarrow y_2(n) = \left(\frac{1}{8}\right)\frac{\left[1-\left(\frac{1}{2}\right)^{n-3}\right]}{1-\frac{1}{2}} = \text{for } n \geq 4$$

$$= \frac{1}{4}\left[1-8\left(\frac{1}{2}\right)^n\right] \text{for } n \geq 4$$

$$= \left[\frac{1}{4}-2\left(\frac{1}{2}\right)^n\right] \text{for } n \geq 4$$

$$= \frac{1}{4}u(n-4) - 2\left(\frac{1}{2}\right)^n u(n-4)$$

$$\Rightarrow h(n) = \left(\frac{1}{2}\right)^n [u(n)-u(n-3)] + 2\left[1-\left(\frac{1}{2}\right)^n\right]u(n-1)$$

$$+\left[\frac{1}{4}-2\left(\frac{1}{2}\right)^n\right]u(n-4)$$

2.13 Correlation of Two Sequences

So far we discussed about the convolution of two signals which is used to find the output $y(n)$ of a system, if the impulse response $h(n)$ of the system and the input signal $x(n)$ are known. In this section, we will study a mathematical operation known as correlation that closely resembles convolution. Correlation is basically used to compare two signals. It occupies a significant place in signal processing. It has application in radar and sonar system where the location of the target is measured by comparing the transmitted and reflected signals. Other

applications of correlation includes in image processing and control engineering etc.

Definition: Correlation is a measure of the degree to which two signals are similar.

The correlation of two signals is divided into (i) Cross-correlation, (ii) Auto-correlation.

2.13.1 Cross-Correlation

The cross-correlation between a pair of signals x (n) and y (n) is given by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \quad l = 0 \pm 1, \pm 2, \pm 3, \qquad \text{.... (2.51)}$$

The index l is the shift (lag) parameter. The order of subscripts xy indicates that x(n) is the reference sequence that remains unshifted in time whereas the sequence $y(n)$ is shifted l units in time with respect to x(n).

If we wish to fix $y(n)$ and to shift x(n), then correlation of two sequences can be written as

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)x(n) \qquad \text{.... (2.52)}$$

If the time shift $l = 0$, then we get

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} y(n)y(n) \qquad \text{... (2.53)}$$

Comparing Eq. (2.51) with Eq. (2.52) we find that

$$\gamma_{xy}(l) = \gamma_{yx}(-l)$$

where $\gamma_{yx}(-l)$ is the folded version of $\gamma_{xy}(l)$ about $l = 0$.

We can rewrite Eq. (2.51) as

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y[-(l-n)]$$

$$= x(l) * y(-l) \qquad \text{... (2.54)}$$

2.13.2 Autocorrelation

The autocorrelation of a sequence is correlation of a sequence with itself. The autocorrelation of a sequence x (n) is defined by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = x(l) * x(-l) \qquad ...(2.55)$$

or equivalently

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)x(n) \qquad ...(2.56)$$

If the time shift $l = 0$, then we have

$$\gamma_{xy}(0) = \sum_{n=-\infty}^{\infty} x^2(n) \qquad ... (2.57)$$

Example 2.14

Determine the cross-correlation sequence $\gamma_{sy}(l)$ of the sequences

$$s(m) = \{2, \underset{\uparrow}{1}, 3\}$$

$$y(n) = \{1, \underset{\uparrow}{2}, 2\}$$

Solution :

Number of sample points in resultant of correlation of two discrete-time sequences $= 3 + 3 - 1 = 5.$

Cross-correlation sequence is defined as

$$\gamma_{SY}(l) = \sum_{n=-\infty}^{\infty} s(n)y(n-l) \qquad l = 0, \pm 1, \pm 2,...$$

For $l = 0$ $\qquad \gamma_{SY}(0) = \sum_{n=-\infty}^{\infty} s(n)y(n)$

$$\boxed{\begin{array}{l} s(n) = \{2, \underset{\uparrow}{1}, 3\} \\ y(n) = \{1, \underset{\uparrow}{2}, 2\} \end{array}}$$

$$\gamma_{SY}(0) = \sum_{n=-\infty}^{\infty} s(n)y(n) = 2 \times 1 + 1 \times 2 + 3 \times 2 = 2 + 2 + 6 = 10$$

For $l = 1$ $\quad \gamma_{SY}(1) = \sum_{n=-\infty}^{\infty} s(n)y(n-1)$

$$\begin{array}{|ll|}\hline s(n) = & \{2, \underset{\uparrow}{1}, 3\} \\ y(n-1) = & \quad\;\; \underset{\uparrow}{1}, 2, 2 \\ \hline \end{array}$$

$$\gamma_{SY}(1) = \sum_{n=-\infty}^{\infty} s(n)y(n-1) = 1 \times 1 + 3 \times 2 = 1 + 6 = 7$$

For $l = 2$ $\quad \gamma_{SY}(2) = \sum_{n=-\infty}^{\infty} s(n)y(n-2)$

$$\begin{array}{|ll|}\hline s(n) = & 2, \underset{\uparrow}{1}, 3 \\ y(n-2) = & \qquad\;\; 1, 2, 2 \\ \hline \end{array}$$

$$\gamma_{SY}(2) = \sum_{n=-\infty}^{\infty} s(n)y(n-2) = 3 \times 1 = 3$$

$$\gamma_{SY}(3) = 0$$
$$\gamma_{SY}(4) = 0$$
$$\gamma_{SY}(5) = 0$$
$$\vdots$$

For $l = -1$, $\quad \gamma_{SY}(-1) = \sum_{n=-\infty}^{\infty} s(n)y(n+1)$

$$\begin{array}{|ll|}\hline s(n) = & \quad 2, \underset{\uparrow}{1}, 3 \\ y(n-1) = & 1, 2, 2 \\ \hline \end{array}$$

$$\gamma_{SY}(-1) = \sum_{n=-\infty}^{\infty} s(n)y(n+1) = 2 \times 2 + 1 \times 2 = 6$$

For $l = -2$, $\quad \gamma_{SY}(-2) = \sum_{n=-\infty}^{\infty} s(n)y(n+2)$

$$\begin{array}{|ll|}\hline s(n) = & \qquad 2, \underset{\uparrow}{1}, 3 \\ y(n+1) = & 1, 2, 2 \\ \hline \end{array}$$

$$\gamma_{SY}(-2) = \sum_{n=-\infty}^{\infty} s(n)y(n+2) = 2 \times 2 = 4$$

$$\gamma_{SY}(-3) = 0$$
$$\gamma_{SY}(-4) = 0$$
$$\gamma_{SY}(-5) = 0$$
$$\vdots$$

The resultant cross-correlation sequence

$$\gamma_{SY}(l) = [\gamma_{SY}(-2), \gamma_{SY}(-1), \gamma_{SY}(0), \gamma_{SY}(1), \gamma_{SY}(2)]$$
$$= \{4, 6, \underset{\uparrow}{10}, 7, 3\}$$

Example 2.15

Compute the auto-correlation of the signal

$$s(n) = A^n u(n),\ 0 < A < 1$$

Solution :

Since s(n) is an infinite-duration signal and its autocorrelation will also have infinite duration. There will be two cases:

Case - I. If $l > 0$

$$\gamma_{ss}(l) = \sum_{n=-\infty}^{\infty} s(n)s(n-l) = \sum_{n=-\infty}^{\infty} A^n u(n).A^{n-l}u(n-l)$$

$$= \sum_{n=l}^{\infty} A^n.A^{n-l}$$

$$= \sum_{n=l}^{\infty} A^n.A^n.A^{n-l} = A^{-l}\sum_{n=l}^{n}\left[A^2\right]^n$$

since A < 1, infinite series coverages

$$= A^{-l}\left[\frac{A^{2l}}{1-A^2}\right] = \frac{A^l}{1-A^2},\ l \geq 0 \qquad ...(a)$$

Ans

Case II. For $l < 0$

$$\gamma_{ss}(l) = \sum_{n=-\infty}^{\infty} s(n)s(n-l) = \sum_{n=0}^{\infty} A^n.A^n.A^{n-l} = A^{n-l}\sum_{n=l}^{n}\left[A^2\right]^n$$

$$Y_{cc}(-l) = \sum_{n=-\infty}^{\infty} s(n).s(n+l) = \sum_{n=-\infty} A^n . A^{n-l} = A^{-l} \sum_{n=l}^{\infty} A^{2n}$$

$$= A^{-l}\cdot\left[\frac{1}{1-A^2}\right] = \frac{A^{-l}}{1-A^2}, l<0 \qquad ....(b)$$

From Eqn. (a) and (b), we get

$$\left.\begin{aligned}\gamma_{ss}(l) &= \frac{A^{l}}{1-A^2}, l\geq 0\\ \gamma_{ss}(l) &= \frac{A^{-l}}{1-A^2}, l<0\end{aligned}\right\}\text{ Auto - correlation sequences}$$

Hence auto-correlation of the signal $s(n) = A^n u(n)$, $0 < A < l$ is given as

$$\gamma_{SS}(l) = \frac{A^{|l|}}{1-A^2}, -\infty < l < \infty$$

2.14 Time Response Analysis of Discrete-time Systems

There are two basic methods for analysing the response of a linear system to a given input signal. In first method the input signal its first resolved into sum of elementary signals (impulse). Then using the linear property of the system the response of the system to the elementary signals are added to obtain the total response.

Second method is based on the direct solution of the difference equation representing the system.

The general form of difference equation is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \qquad ... (2.58)$$

where N is called the order of the difference equation. The solution of the difference equation consists of two parts i.e.,

where $y_h(n)$, the natural response is known as the homogenous solution and $y_p(n)$ the forced response is called as particular solution.

The homogenous solution is obtained by setting terms involving the input x(n) to zero. Thus from Eq. (2.58) we have

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \qquad ... (2.59)$$

where $a_0 = 1$

To solve the Eq. (2.59) assume

$$y_h(n) = \lambda^n \qquad ... (2.60)$$

where the subscript h on y(n) is used to denote the solution to the homogeneous difference equation.

Substituting Eq. (2.60) in Eq. (2.59) we get

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

$$\lambda^{n-N}[\lambda^N + a_1\lambda^{N-1} + a_{N-1}\lambda + a_N] = 0$$

which gives

$$\lambda^N a_1 \lambda^{N-1} + ...a_{N-1}\lambda + a_N = 0 \quad ...(2.61)$$

The Eq. (2.61) is known as characteristic equation and has N roots, which we denote as

$\lambda_1, \lambda_2 ...., \lambda_N$

If $\lambda_1, \lambda_2 ...., \lambda_N$ are distinct, the general solution is of the form

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + ...C_N\lambda_N^n \quad ...(2.62)$$

For example, if the roots are $\lambda_1 = 2$ and $\lambda_2 = 3$, then

$$y_h(n) = C_1(2)^n + C_2(3)^n \quad ...(2.63)$$

If the roots of the characteristic equation are repeated, say λ_1 is repeated for m times, then the general solution of $y_h(n)$ contains the term

$$\boxed{\lambda_1^n(C_1 + C_2 n + C_3 n^2 + ...C_m n^{m-1})} \quad ...(2.64)$$

For each repeat root, there is a term of this form in $y_h(n)$.

If $\lambda_1 = 2$ is repeated for 2 times then $2^n(C_1 + nC_2)$ is the general solution.

If the characteristic equation has complex roots for example,

$$\lambda_1, \lambda_2 = a \pm jb$$

then the solution $y_h(n) = r^n (A_1 \cos n\theta + A_2 \sin n\theta)$ ...(2.65)

where $r = \sqrt{a^2 + b^2}$ ...(2.66)

$\theta = \tan^{-1} b/a$ ...(2.67)

A_1 and A_2 are constants.

The particular solution $y_p(n)$ is to satisfy the difference equation for the specific input signal x(n), $n \geq 0$. In other words, $y_p(n)$ is any solution satisfying

$$1 + \sum_{k=1}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad ...(2.68)$$

To solve Eq. (2.68), we assume for $y_p(n)$, a form that depends on the form of x(n). The general form of the particular solution for several inputs are shown in table 2.1.

Table 2.1. General *form of particular* solution *for several types of input*

$x(n)$ input signal	$y_p(n)$ Particular solution
A (Step input)	K
AM^n	KM^n
An^M	$K_0n^M + K_1n^{M-1} \ldots K_M$
$A^n n^M$	$A^n(K_0n^M + K_1n^{M-1} + \ldots K_M)$
$\left.\begin{array}{l} A\cos\omega_0 n \\ A\sin\omega_0 n \end{array}\right\}$	$K_1\cos\omega_0 n + K_2\sin\omega_0 n$

To obtain the total solution we have to add the homogeneous solution and particular solution. Thus

$$y(n) = y_h(n) + y_p(n) \quad \ldots (2.69)$$

The resultant sum $y(n)$ contains the constant parameters $\{y_i\}$ embodied in the homogeneous solution component $y_h(n)$. These constants can be determined by applying initial conditions.

2.14.1 Impulse Response

The general form of difference equation is

$$y(n) = \sum_{k=1}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad \ldots (2.70)$$

For the input $x(n) = \delta(n)$

$$\sum_{k=0}^{M} b_k x(n-k) = 0 \quad \text{for} \quad n > M \quad \ldots (2.71)$$

Then Eq. (2.70) can be written as

$$\boxed{y(n) = \sum_{k=0}^{N} a_k y(n-k) = 0 \qquad a_0 = 1} \quad \ldots (2.72)$$

The solution of Eq.(2.72) is known as homogeneous solution. The particular solution is zero since $x(n) = 0$ for $n > 0$, that is

$$y_0(n) = 0 \quad \ldots (2.73)$$

Therefore we can obtain the impulse response by solving the homogenous equation and imposing the initial conditions to determine the arbitrary constants.

2.14.2 Step response

The step response can be easily expressed in terms of the impulse response using convolution sum. Let a discrete time system have impulse response h(n) and denote the step response as s(n).

The s(n) = h(n) * u(n)

$$= \sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

Since $u(n-k) = 0$ for $k > n$ and $u(n-k) = 1$ for $k \leq n$ we have

$$s(n) \sum_{k=-\infty}^{\infty} h(k) \qquad \text{... (2.74)}$$

That is, the step response is the running sum of the impulse response.

Example 2.16

The discrete-time system

$$y(n) = ny(n-1) + x(n), \qquad n \geq 0$$

is at rest [i.e., $y(-1) = 0$]. Check if the system is linear time invariant and BIBO stable.

Solution :

$$y(n) = ny(n-1) + x(n) \qquad \text{....(I)}$$

The solution for $y(n) = y_h(n) + y_p(n)$

$y_h(n) \Rightarrow$ homogenous solution

$y_p(n) \rightarrow$ particular solution

Hence we have to find impulse response

i.e. $x(n) = \delta(n)$

so $y_p(n) = 0$

Let $y_h(n) = \lambda^n$

So $\lambda^n = n\lambda^{n-1}$ ($\because x(n) = 0$ for homogenous solution)

$\Rightarrow \quad n\lambda^{n-1}(\lambda - n) = 0$

$\Rightarrow \quad \lambda = 0$ or n

So $y_h(n) = An^n$

$y(n-k) = A(n-k)^{n-k}$ ...(II)

$y(n, k) = A(n^n - k)$

$y(n, k) \neq y(n-k) \rightarrow$ Time variant

$$\sum_{n=-\infty}^{\infty} y(n) = A\sum_{n=-\infty}^{\infty} n^n = \infty$$

It is unstable.

Example 2.17

Determine the zero-input response of the system described by the second-order difference equation.

$$x(n) - 3y(n-1) - 4y(n-2) = 0$$

Solution :

$$x(n) - 3y(n-1) - 4y(n-2) = 0$$

For zero input response, i.e. $x(n) = 0$,

Also, $-3y(n-1) - 4y(n-2) = 0$

$$\Rightarrow \quad y(n-1) = \frac{-4}{3} y(n-2)$$

$$\Rightarrow \quad y(-1) = \frac{-4}{3} y(-2) \qquad (\because \text{For } n = 0)$$

For $n = 1$, $y(0) = \frac{-4}{3} y(-1) = \left(\frac{-4}{3}\right)^2 y(-2)$

$$\Rightarrow \quad \text{Solution is } \boxed{y(n) = \left(\frac{-4}{3}\right)^{n+2} y(-2)}$$

Example 2.18

Determine the impulse response of the following causal system :

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Solution :

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1) \qquad \text{...(I)}$$

For impulse response the particular solution is zero.

Now for homogenous solution,

$y(n) - 3y(n-1) - 4y(n-2) = 0$

Let $\quad y(n) = \lambda^n$

so $\quad \lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$

$\Rightarrow \quad \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow \quad \lambda = 4, -1$

so $\quad y_h(n) = C_1 4^n + C_2(-1)^n \qquad \text{...(II)}$

$\delta(0) = 1 \Rightarrow \boxed{x(0) = 1}$

For $n = 0$, $\quad y(0) - 3y(-1) - 4y(-2) = x(0) + 2x(-1)$ from eqn. (I)

$\Rightarrow \quad y(0) = 1$

$$y(0) = C_1 + C_2 \qquad \text{From equation(II)}$$

$$\Rightarrow \quad C_1 + C_2 = 1 \qquad \text{....(III)}$$

For n = 1, $\quad y(1) - 3y(0) - 0 = 0 + 2.1$

$$\Rightarrow \quad y(1) = 5 \qquad \text{From equation(I)}$$

$$y(1) = 4C_1 - C_2 \qquad \text{From equation(II)}$$

$$\Rightarrow \quad 4C_1 - C_2 = 5 \qquad \text{....(IV)}$$

From eqns (III) and (IV),

$$C_1 = \frac{6}{5}$$

$$\& \quad C_2 = \frac{-1}{5}$$

So $\quad y(n) = \frac{6}{5}(4)^n - \frac{1}{5}(-1)^n$

Example 2.19

Determine the response of the system with impulse response

$$h(n) = a^n u(n)$$

to the input signal

$$x(n) = u(n) - u(n-10)$$

Solution :

$$h(n) = a^n u(n)$$

$$x(n) = u(n) - u(n-10)$$

$$y(n) = x(n) * h(n)$$

$$= u(n) * a^n u(n) - u(n-10) * u(n) \,.\, a^n$$

$$= \sum_{n=-\infty}^{\infty} u(k) * a^{n-k} u(n-k) - \sum_{k=-\infty}^{\infty} u(k-10) u(n-k) a^{n-k}$$

$$= \sum_{k=0}^{n} a^n . a^{-K} - \sum_{k=10}^{n} a^n . a^{-K}$$

$$= a^n \left[\frac{1 - \left(\frac{1}{a}\right)^{n+1}}{1 - \frac{1}{a}} - a^{-10} \frac{1 - \left(\frac{1}{a}\right)^{n-9}}{1 - \frac{1}{a}} \right]$$

Example 2.20

Determine the impulse response and the unit step response of the systems described by the difference equation

$y(n) = 0.6\, y(n-1) - 0.08\, y(n-2) + x(n)$

Solution :

$$y(n) = 0.6\, y(n-1) - 0.08\, y(n-2) + x(n) \qquad \text{...(I)}$$

Here solution of $y(1) = y_h(n) + y_p(n)$

For unit step response i.e. $x(n) =$ u(n)

$y_p(n) = K$

So from equation (I),

$K = 0.6\, K - 0.08\, K + 1$

$\Rightarrow 0.48\, K = 1$

$\Rightarrow K = 1/0.48 \simeq 2$

$$\Rightarrow \boxed{y_P(n) = 2} \qquad \text{...(II)}$$

For homogenous solution, equation(1) becomes,

$y(n) - 0.6\, y(n-1) + 0.08\, y(n-2) = 0$

Let $y(n) = \lambda^n$

So $\lambda^n - 0.6\, \lambda^{n-1} + 0.08\, \lambda^{n-2} = 0$

$\Rightarrow \quad \lambda^2 - 0.6\lambda + 0.08 = 0$

$\Rightarrow \quad 100\lambda^2 - 60\, \lambda + 8 = 0$

$\Rightarrow \quad 25\lambda^2 - 15\, \lambda + 2 = 0$

$\Rightarrow \quad 5(\lambda - 2)(5\lambda - 1) = 0 \Rightarrow \lambda = \frac{1}{5} \text{ or } \frac{2}{5}$

$$\text{So } y_h(n) = C_1\left(\frac{1}{5}\right)^n + C_2\left(\frac{2}{5}\right)^n \qquad \text{...(III)}$$

$$\text{So } y(n) = C_1\left(\frac{1}{5}\right)^n + C_2\left(\frac{2}{5}\right)^n + 2 \qquad \text{...(IV)}$$

$y(0) = C_1 + C_2 + 2$

$y(1) = \frac{1}{5} C_1 + \frac{2}{5} C_2 + 2$

From equation (I),

$$y(0) = x(0) = 1$$

$$y(1) = 0.6\,y(0) + x(0)$$

$$= 1.6$$

Now $C_1 + C_2 + 2 = 1$

$$\Rightarrow \quad C_1 + C_2 + -1 \qquad \text{...(V)}$$

$$\& \quad \frac{C_1}{5} + \frac{2C_2}{5} + 2 = 1.6$$

$$\Rightarrow \quad C_1 + 2C_2 + 10 = 8$$

$$\Rightarrow \quad C_1 + 2C_2 = -2 \qquad \text{...(VI)}$$

Solving equation (V) & (VI),

$$C_2 = -1, C_1 = 0$$

So $\boxed{y(n) = -\left(\frac{2}{5}\right)^n + 2}$

MISCELLANEOUS SOLVED EXAMPLES

Example 2.21

A discrete-time signal x(n) is defined as

$$x(n) = \begin{cases} 1 + \frac{n}{3}, & -3 \le n \le -1 \\ 1, & 0 \le n \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine its values and sketch the signal x(n).

(b) Can you express the signal x(n) in terms of signals δ(n) and u(n) ?

Solution :

$$x(n) = \begin{cases} 1 + \frac{n}{3} & -3 \le n \le -1 \\ 1 & 0 \le n \le 3 \\ 0 & \text{else where} \end{cases}$$

(a) $x(n) = \{0, -0.33, -0.67, \underset{\uparrow}{1}, 1, 1, 1\}$

(b) $x(n) = (-0.33)\,\delta(n+2) - 0.67\,\delta(n+1) +$ 1 , $\delta(n)$

$$+\ \delta(n-1)+\delta(n-2)+\delta(n-3)$$

$$\begin{aligned}
\text{Again } x(n) = \; & -0.33\,[u(n+2)-u(n+1)] \\
& -0.67\,[u(n+1)-4(n)\,] \\
& +[u(n)-u(n-1)]\cdot 1 \\
& +[u(n-1)-u(n-2)]\cdot 1 \\
& +[u(n-2)-u(n-3)]\cdot 1 \\
& +[u(n-3)-u(n-4)]\cdot 1
\end{aligned}$$

$$\Rightarrow \quad x(n) = -0.33\ u(n+2) - u(n+1) + 1.67\ u(n) - u(n-4)$$

Example 2.22

i) Show that any signal can be decomposed into an even and an odd component.

ii) Is the decomposition unique

iii) Illustrate your arguments using the signal

$$x(n) = \{2, 3, \underset{\uparrow}{4}, 5, 6\}$$

Solution :

(i) Let a signal is x(n)

Then its inverted signal is x(−n)

The even part of the signal is;

$$x_e(n) = \frac{x(n)+x(-n)}{2}$$

The odd part of the signal is,

$$x_0(n) = \frac{x(n)-x(-n)}{2}$$

(ii) Yes the decomposition is unique

(iii) $x(n) = \{2, 3, \underset{\uparrow}{4}, 5, 6\}$

$$x(-n) = \{5, 6, \underset{\uparrow}{4}, 3, 2\}$$

$$x_e(n) = \frac{x(n)+x(-n)}{2} = \{3.5, 4.5, \underset{\uparrow}{4}, 4, 4\}$$

$$x_0(n) = \frac{x(n)-x(-n)}{2} = \{-1.5, -1.5, \underset{\uparrow}{0}, 1, 2\}$$

Example 2.23

Consider the system

$$y(n) = T[x(n)] = x(n^2)$$

Determine if the system is time invariant.

Solution :

$y(n) = x(n^2)$

$$y(n, k) = T[x(n-k)]$$
$$= x(n^2 - k)$$
$$y(n-k) = x((n-k)^2)$$
$$= x(n^2 + k^2 - 2nk)$$

Here $y(n, k) \neq y(n-k)$

so it is time variant

Example 2.24

Compute the convolution of following signal.

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, -3\}, \quad h(n) = \{\underset{\uparrow}{1}, 0, -1, -1\}$$

Solution :

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, -3\}, \quad h(n) = \{\underset{\uparrow}{1}, 0, -1, -1\}$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$y(n)$ will start from $n = 0$.

Total no. of signals in $y(n) = 4 + 4 - 1 = 7$

i.e. $0 \leq n \leq 6$.

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= x(0)\, h(0) = 0$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

$$= x(0)\,h(1) + x(1)\,h(0) = 0 + 1 = 1$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0)$$

$$= 0 + 0 + 4 = 4$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h\,(0)$$

$$= 0 + (-1) + 0 + (-3)\,1$$

$$= -4$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= x(0)\,h(4) + x(1)\,h(3) + x(2)\,h(2) + x(3)\,h(1)$$

$$= 0 + 1\,(-1) + 4.\,(-1) + (-3)\,.\,0$$

$$= -5$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$$

$$= x(2)\,h(3) + x(3)\,h(2)$$

$$= 4.\,(-1) + (-3)\,.\,(-1)$$

$$= -1$$

$$y(6) = \sum_{k=-\infty}^{\infty} x(k)h(6-k)$$

$$= x(3)\,h(3)$$

$$= (-3)\,.\,(-1)$$

$$= 3$$

so $y(n) = \{\underset{\uparrow}{0}, 1, 4, -4, -5, -1, 3\}$

Example 2.25

Compute the convolution of following pair of signals.

$$x(n) = u(n+1) - u(n-4) - \delta(n-5)$$

$$h(n) = [u(n+2) - u(n-3)].(3 - |n|)$$

Solution :

$$x(n) = 4(n+1) - u(n-4) - \delta(n-5)$$

$$= \{1, \underset{\uparrow}{1}, 1, 1, 1, 0, -1\} \quad n_1 = -1, N_1 = 7$$

$$h(n) = [u(n+2) - u(n-3)].(3 - |n|)$$

$$= \{1, 2, \underset{\uparrow}{3}, 2, 1\} \qquad n_2 = -2, \; N_2 = 5$$

$$y(n) = x(n) * h(1) \text{ will start at } n = n_1 + n_2 = -3$$

Total no. of values of y(n) is $7 + 5 - 1 = 11$

i.e $\quad -3 \le n \le 7$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(-3) = \sum_{k=-\infty}^{\infty} x(k)\, h(-3-k)$$

$$= x(-1)\, h(-2) = 1$$

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)\, h(-2-k)$$

$$= x(-1)\, h(-1) + x(0)\, h(-2)$$

$$= 2 + 1 = 3$$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

$$= x(-1)\, h(0) + x(0)\, h(-1) + x(1)\, h(-2)$$

$$= 3 + 2 + 1 = 6$$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k)$$

$$= x(-1)\, h(1) + x(0)\, h(0) + x(1)\, h(-1) + x(2)\, h(-2)$$

$= 1.2 + 1.3 + 1.2 + 1.1$

$= 8$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\ h(1-k)$$

$= x(-1)\ h(2) + x(0)\ h(1) + x(1)\ h(0) + x(2)\ h\ (-1) + x(3)\ h\ (-2)$

$= 1.1 + 1.2 + 1.3 + 1.2 + 1.1$

$= 9$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\ h(2-k)$$

$= x(0)\ h(2) + x(1)\ h(1) + x(2)\ h(0) + x(3)\ h(-1) + x(4)\ h(-2)$

$= 1.1 + 1.2 + 1.3 + 1.2 + 0$

$= 8$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\ h(3-k)$$

$= x(1)\ h(2) + x(2)\ h(1) + x(3)\ h(1) + x(4)\ h\ (-1) + x(5)\ h(-2)$

$= 1.1 + 1.2 + 1.3 + 0 + (-1) = 7$

$= 5$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\ h(4-k)$$

$= x(2)\ h(2) + x(3)\ h(1) + x(4)\ h(0) + x(5)\ h(-1)$

$= 1.1 + 1.2 + 0 + (-1)\ .2$

$= 1$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\ h(5-k)$$

$= x(3)\ h(2) + x(4)\ h(1) + x(5).\ h(0)$

$= 1.1 + 0 + (-1).\ 3$

$= -2$

$$y(6) = \sum_{k=-\infty}^{\infty} x(k)\ h(6-k)$$

$$= x(4)\,h(2) + x(5)h(1)$$

$$= 0 + (-1).\ 2 = -2$$

$$y(7) = \sum_{k=-\infty}^{\infty} x(k)\ h(7-k)$$

$$= x(5)\ h(2) = (-1)\ .\ 1 = -1$$

$$\text{So } y(n) = \{1, 3, 6, \underset{\uparrow}{8}, 9, 8, 5, 1, -2, -1\}$$

Example 2.26

Check whether the systems described by the following equations are causal:

(i) $y(n) = 3x(n-2) + 3x(n+2)$

(ii) $y(n) = x(n-1) + ax(n-2)$

(iii) $y(n) = x(-n)$.

Solution : The given expression is

$$y(n) = 3x(n-2) + 3x(n+2)$$

From above equation, it is clear that y(n) is determined using the past input sample value $3x(n-2)$ and future input sample value $3x(n+2)$.

Therefore, the given system is a non-causal system.

(ii) The given system is

$$y(n) = x(n-1) + ax(n-2)$$

From this equation, it is clear that y(n) is determined using that y(n) is determined using only the previous input sample values $x(n-1)$ and $ax(n-2)$.

Therefore, the given system is a causal system.

(iii) The given system is

$$y(n) = x(-n)$$

From this equation, it is clear that the input sample value is located on the negative time axis and the sample values cannot be obtained before $t = 0$.

Therefore, the given system is a non-causal system.

Example 2.27

A discrete-time system is represented by the following difference equation in which x(n) is input and y(n) is the output:

$$y(n) = 3y^2\,(n-1) - nx(n) + 4x(n-1) - 2x(n+1)$$

Is this system Linear ? Shift-invariant ? Causal ?

In each case, justify your answer.

Solution:

(i) Check for the linearity

The given expression is

$$y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - 2x(n+1)$$

It may be noted that the real condition for linearity is

$$F[ax(n)] = a.\,F[x(n)]$$

Now, $F[ax(n)] = ay(n) = \quad 3a^2y^2(n-1) - anx(n) + 4ax(n-1) - 2ax(n+1)$

and $a.F[x(n)] = a[y(n)] = \quad 3ay^2(n-1) - anx(n) + 4ax(n-1) - 2ax(n+1)$

From above, it is clear that

$$F[ax(n)] \neq a.F[x(n)]$$

Therefore, the system is non-linear.

(ii) Check for Shift invariant.

The given system is

$$y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - 2x\,(n+1)$$

It may be noted that the necessary condition for shift-invariance is

$$y(n-k) = F[x(n-k)]$$

Now, $F[x(n-k)] = 3y^2(n-k-1) - nx(n-k) + 4x(n-k-1) - 2x(n-k+1).$

Also,

$$y(n-k) = 3y^2(n-k-1) - (n-k).\,x(n-k) + 4x(n-k-1) - 2x(n-k+1).$$

Since $y(n-k) \neq F.[x(n-k)]$

Therefore, the given system is ~~time-invariant~~. time variant.

(iii) Check for the Causality : The given system is

$$y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - 2x(n+1)$$

It may be noted that the required condition for causality is that the output of a causal system must be dependent only on the present and past values of the input.

From the given equation, it is obvious that the output y(n) is dependent on a future input sample value x(n + 1).

Therefore, the given system is a non-causal system.

Example 2.28

Check about linearity of the following systems :

(i) $F[x(n)] = an.x(n) + b$

(ii) $F[x(n)] = e^{x(n)}$

imp

Solution: (i) The given expression is

$$F[x(n)] = anx(n) + b$$

Now, for two values, $x_1(n)$ and $x_2(n)$, we have

$$F[x_1(n) + x_2(n)] = an[x_1(n) + x_2(n)] + b$$

or $$F[x_1(n)] + F[x_2(n)] = [anx_1(n) + b] + [an\, x_2(n) + b]$$

Now, since from equation (i), it is evident that

$$F[x_1(n) + x_2(n)] \neq F[x_1(n)] + F[x_2(n)]$$ **Ans**

therefore, the given system is non-linear when $b \neq 0$

(ii) The given expression is

$$F[x(n)] = e^{x(n)}$$

For two values, $x_1(n)$ and $x_2(n)$, we have

$$F[x_1(n) + x_2(n)] = e^{x_1(n) + x_2(n)} = e^{x_1(n)}.e^{x_2(n)} \quad ...(i)$$

or $$F[x_1(n)] + F[x_2(n)] = e^{x_1(n) + x_2(n)}$$

Now, since from equation (i), it is evident that

$$F[x_1(n) + x_2(n)] \neq F[x_1(n)] + F[x_2(n)]$$ **Ans.**

Therefore the system is not linear

Example 2.29

Test the following systems for linearity.

(i) $y(nT) = F[x(nT)] = 9x^2(nT - T)$

(ii) $y(nT) = F[x(nT)] = (nT)^3.\, x(nT + 22)$.

in Linearity

$T[x(n)]$

$T[x_1(n) + x_2(n)]$

Solution : (i) Given expression is

$$y(nT) = F[x(nT)] = ax^2(nT - T)$$

For a constant, a other than unity, we have

$$F[ax(nT)] = 9a^2\, x^2(nT - T)$$

and $$aF[x(nT)] = 9ax^2(nT - T)$$

Here, since, $F[ax(nT)] \neq a.F[x(nT)]$,

therefore, the given system is not linear.

(ii) The given system is

$$y(nT) = F[x(nT)] = (nT)^3.\, x(nT + 27)$$

For two values, $x_1(nT)$ and $x_2(nT)$, we have

$$F[ax_1(nT) + bx_2(nT)] = (nT)^2[ax_1(nT + 2T) + bx_2(nT + 2T)]$$

or $$F[ax_1(nT) + bx_2(nT)] = a(nT)^2\, x_2(nT + 2T) + b(nT)^2\, x_2(nT + 2T)$$

or $$F[ax_1(nT) + bx_2(nT)] = a.F[x_1(nT)] + bF[x_2(nT)] \quad ...(i)$$

From equation (i), it is evident that the given system is linear.

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Example 2.30

Check whether the systems described by the following equations are time-invariant or time-variant:

(i) $y(n) = F[x(n)] = an.x(n)$

(ii) $y(n) = F[x(n)] = ax(n-1) + bx(n-2)$.

Solution : (i) The given expression is

$$y(n) = F[x(n)] = an.x(n)$$

Now, the response to a delayed excitation is given by

$$F[x(n-k)] = an.[x(n-k)] \quad ...(i)$$

and the delayed response is

$$y(n-k) = a(n-k)\,[x(n-k)] \quad ...(ii)$$

Here, from equations (i) and (ii), it may be observed that

$$F[x(n-k)] \neq y(n-k)$$

Therefore, the system is not time-invariant, i.e., the system is time dependent. (Time variant)

(ii) The given expression is

$$y(n) = F[x(n)] = ax(n-1) + bx(n-2)$$

Here, the response to a delayed excitation is given by

$$F[x(n-k)] = ax[(n-k)-1] + bx[(n-k)-2] = y(n-k)$$

$$= \text{The delayed response}$$

Thus, in this case, we have

$$Flx(n-k)] = y(n-k)$$

and therefore, the given system is a time-invariant system.

Example 2.31

Test whether the system described by the equation

$$F[x(n)] = n[x(n)]^2$$

is linear and time-invariant.

Solution : Check for the linearity :

The given system is

$$F[x(n)\} = n(x(n)\}^2$$

For two values, $x_1(n)$ and $x_2(n)$, we have

$$F[x_1(n)] = n[x_1(n)]^2$$

and $$F[x_2(n)] = n[x_1(n)]^2$$

Therefore,

$$F[x_1(n)] + F[x_2(n)] = n[\{x_1(n)\}^2 + \{x_2(n)\}^2]$$

Further, we have

$$F[x_1(n) + x_2(n)] = n[x_1(n) + x_2(n)]^2$$
$$= n[\{x_1(n)\}^2 + \{x_2(n)\}^2 + 2x_1(n)x_2(n)\}]$$

Here, since

$$F[x_1(n) + x_2(n)] \neq F[x_1(n)] + F[x_2(n)]$$

Therefore, the given system is a non-linear system.

Check for Time-Invariant :

The given expression is

$$F[x(n)] = n[x(n)]^2 = y(n)$$

Now, the response to a delayed excitation is

$$F[(x(n-k)] = n[(x(n-k)]^2$$

Also, the delayed response will be

$$y(n-k) - (n-k)[x(n-k)]^2.$$

Thus, we observe that

$$y(n-k) \neq F[x(n-k)]$$

Therefore, the given system is not a time-invariant system. (Time variant system.)

Example 2.32

Check the discrete-time system for time-invariance which is described by the following difference equation

$$y(n) = 4n\,x(n)$$

Solution: The response to a delayed input is

$$y(n, k) = 4n\,x(n-k) \quad \text{...(i)}$$

The delayed response will be

$$y(n-k) = 4(n-k)\,x(n-k) \quad \text{...(ii)}$$

It is clear that both responses are not equal, i.e.

$$y(n, k) \neq y(n-k)$$

Therefore, the given discrete-time system $y(n) = 4n\,x(n)$ is not time-invariant. It is a time-varying system.

Example 2.33

Test whether the system described by the equation

$$F[x(n)] = a[x(n)]^2 + b.x(n)$$

is linear and time-invariant.

Solution : Test for linearity :

The given system is

$$F[x(n)] = a[x(n)]^2 + b.x(n)$$

For two values of $x_1(n)$ and $x_2(n)$, we have

$$F[x_1(n)] = a[x_1(n)]^2 + bx_1(n)$$

and $$F[x_2(n)] = a[x_2(n)]^2 + bx_2(n)$$

Therefore,

$$F[x_1(n)] + F[x_2(n)] = a[\{x_1(n)\}^2 + \{x_2(n)\}^2] + b[x_1(n) + x_2(n)] \quad ...(i)$$

Also $F[x_1(n) + x_2(n)] = a[x_1(n) + x_2(n)]^2 + b[x_1(n) + x_2(n)]$

or $$F[x_1(n) + x_2(n)] = a[\{x_1(n)\}^2 + \{x_2(n)\}^2 + 2x_1(n)x_2(n)] + bx_1(n) + bx_2(n) \quad ...(ii)$$

From equations (i) and (ii), it is clear that

$$F[x_1(n) + x_2(n)] \neq F[x_1(n)] + F[x_2(n)]$$

Therefore, the given system is a non-linear system.

Test for Time-invariant :

The given system is

$$F[x_1(n)] = a[x(n)]^2 + bx(n) = y(n)$$

Now, the response to a delayed excitation is

$$F[x(n-k)] = a[x(n-k)]^2 + bx(n-k) \quad ...(iii)$$

and the delayed response is

$$y(n-k) = a[x(n-k)]^2 + b[x(n-k)] \quad ...(iv)$$

From equations (iii) and (iv), it is clear that the system is time-invariant.

Example 2.34

The input x(n) and the impulse response h(n) of a discrete-time LTI system are given by

$$x(n) = u(n) \text{ and } h(n) = a^n u(n) \quad 0 < a < 1$$

(a) Compute the output, y(n) by equation

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad ...(i)$$

(b) Compute the output y(n) by equation

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad ...(ii)$$

Solution : By equation (i), we have

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Sequences x(k) and h(n – k) are shown in figure 2.31(a) for $n < 0$ and $n > 0$. From figure 2.31(a) we observe that for $n < 0$, x(k) and h(n – k) do not overlap, while for $n \geq 0$, they overlap from $k = 0$ to $k = n$. Hence, for $n < 0$, $y(n) = 0$. For $n \geq 0$, we have

$$y(n) = \sum_{k=0}^{\infty} a^{n-k}$$

Changing the variable of summation k to $m = n - k$ and using equation (i), we have

$$y(n) = \sum_{m=n}^{0} \alpha^{m} = \sum_{m=0}^{n} \alpha^{m} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

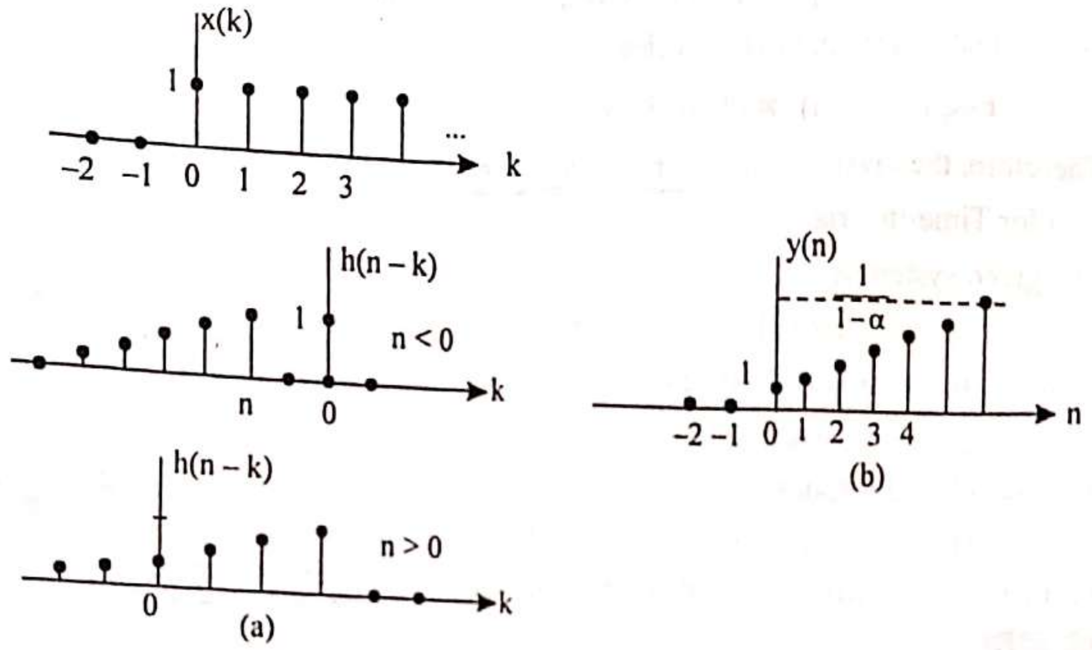

Fig. 2.31

Thus, we can write the output y(n) as under :

u(n) * $a^n \cdot u(n)$

$$\boxed{y(n) = \left(\frac{1-\alpha^{n-1}}{1-\alpha}\right)u(n)} \qquad \text{....(iii)}$$

which has been sketched in figure 2.31(b)

(b) By equation (ii), we get

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Sequences h(k) and x(n − k) are shown in figure 2.32, for $n < 0$ and $n > 0$. Again from figure 2.32 we see that for $n < 0$, h(k) and x(n − k) do not overlap, while for $n > 0$, they overlap from $k = 0$ to $k = n$. Hence, for $n < 0$, $y(n) = 0$. For $y(n) > 0$, we have

$$y(n) = \sum_{k=0}^{n} \alpha^{k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Thus, we obtain the same result as shown in equation (iii).

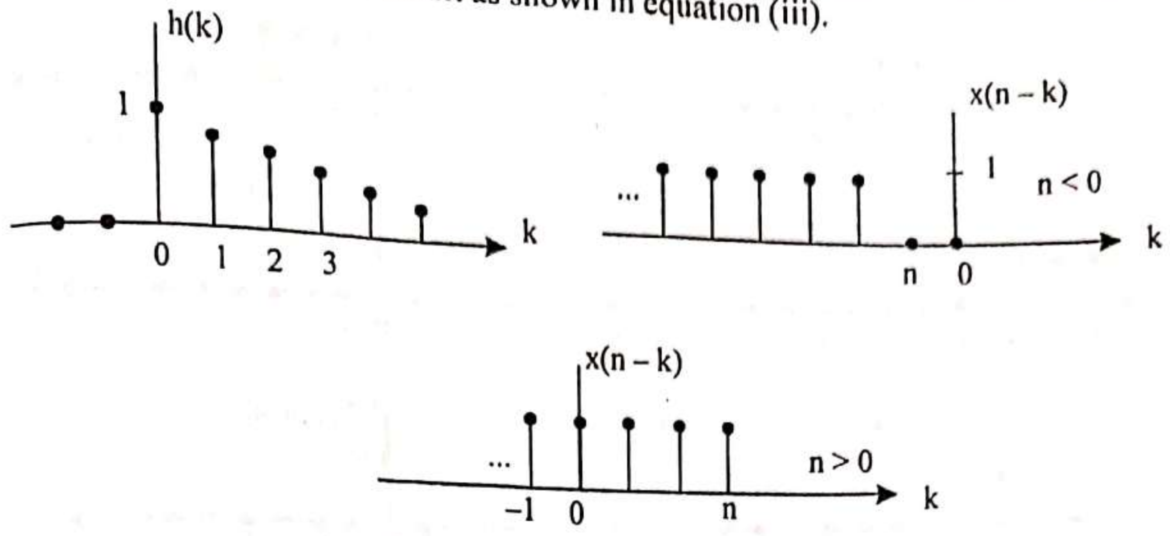

Fig. 2.32

Example 2.35

Evaluate $y(n) = x(n) * h(n)$, where $x(n)$ and $h(n)$ are shown in figure 2.33 by an analytical technique

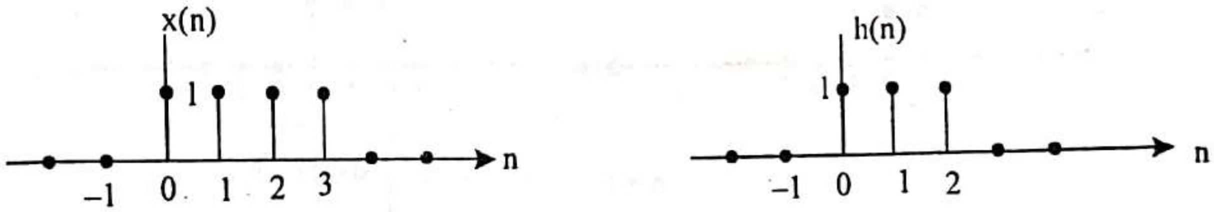

Fig. 2.33

Solution : Note that $x(n)$ and $h(n)$ can be expressed as under :

$$x(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$$
$$h(n) = \delta(n) + \delta(n-1) + \delta(n-2)$$

Now, we have

$$x(n) * h(n) = x(n) * \{\delta(n) + \delta(n-1) + \delta(n-2)\}$$
$$= x(n) * \delta(n) + x(n) * \delta(n-1) + x(n) * \delta(n-2)\}$$
$$= x(n) + x(n-1) + x(n-2)$$

Thus, $y(n) = \delta(n) + 5(n-1) + \delta(n-2) + \delta(n-3)$
$$+ \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)$$
$$+ \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)$$

or $y(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 3\delta(n-3) + 2\,\delta(n-4) + \delta(n-5)$

or $y(n) = \{\underset{\uparrow}{1}, 2, 3, 3, 2, 1\}$

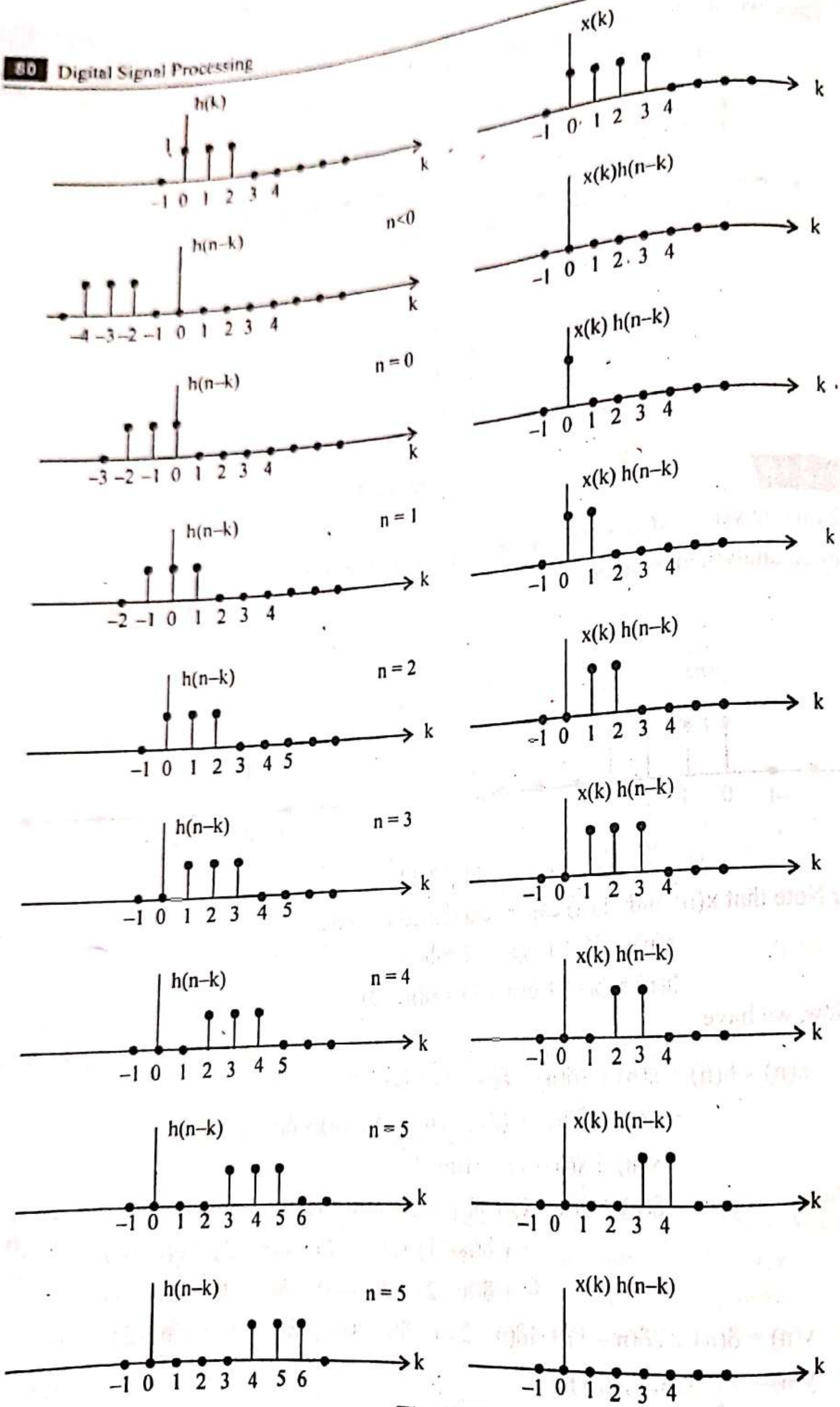

Fig. 2.34

Example 2.36

Show that if the input x(n) to discrete-time LTI system is periodic with period N, then the output y(n) is also periodic with period No.

Solution : Let h(n) be the impulse response of the system. Then, we have.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Let $\quad n = m + N.$

Then $\quad y(m+N) = \sum_{k=-\infty}^{\infty} h(k)x(m+N-k) = \sum_{k=-\infty}^{\infty} h(k)x((m-k)+N)$

Since x(n) is period with period N. we have

$$x[(m-k)+N] = x(m-k)$$

Thus , $\quad y(m+N) = \sum_{k=-\infty}^{\infty} h(k)x(m-k) = y(m)$

which, indicates that the output y(n) is periodic with period N.

Example 2.37

Find the impulse response h(n) for each of the causal LTI discrete-time systems satisfying the following difference equations and state whether each systems a FIR or an IIR system.

(a) $\quad y(n) = x(n) - 2x(n-2) + x(n-3)$

(b) $\quad y(n) + 2y(n-1) = x(n) + x(n-1)$

(c) $\quad y(n) - \frac{1}{2}y(n-2) = 2x(n) - x(n-2)$

(different model)

$x(n) \rightarrow \delta(n)$

$y(n) \rightarrow h(n)$

Solution : (a) By definition, we have

$$h(n) = \delta(n) - 2\delta(n-2) + \delta(n-3)$$

or $\quad h(n) = \{1, 0, -2, 1\}$

Since h(n) has only four terms, the system is a FIR system.

(b) $\quad h(n) = -2h(n-1) + \delta(n) + \delta(n-1)$

Since the system is causal, h(−1) = 0. Then

$$h(0) = -2h(-1) + \delta(0) + \delta(-1) = \delta(0) = 1$$

$$h(1) = -2h(0) + \delta(1) + \delta(0) = -2 + 1 = -1$$

$$h(2) = -2h(1) + \delta(2) + \delta(1) = -2(-1) = 2$$

$$h(3) = -2h(2) + \delta(3) + \delta(2) = -2(2) = -2^2$$

⋮

$$h(n) = -2h(h-1) + \delta(n) + \delta(n-1) = (-1)^n 2^{n-1}$$

Hence, $h(n) = \delta(n) + (-1)^n 2^{n-1} u(n-1)$

Since h(n) has infinite terms, therefore the system is an IIR system.

(c) $h(n) = \frac{1}{2} h(n-2) + 2\delta(n) - \delta(n-2)$

Since the system is causal, $h(-2) = h(-1) = 0$.

Then $h(0) = \frac{1}{2} h(-2) + 2\delta(0) - \delta(-2) = 2\,\delta(0) = 2$

$$h(1) = \frac{1}{2} h(-1) + 2\delta(1) - \delta(-1) = 0$$

$$h(2) = \frac{1}{2} h(0) + 2\delta(2) - \delta(0) = \frac{1}{2}(2) - 1 = 0$$

$$h(3) = \frac{1}{2} h(1) + 2\delta(3) - \delta(1) = 0$$

⋮

Hence, h(n) $h(n) = 2\delta(n)$

Since h(n) has only one term, therefore, the system is a FIR system.

Example 2.38

Test if the following systems are stable or not. imp

(i) $y(n) = \cos x(n)$ (ii) $y(n) = \sum_{k=-\infty}^{n+1} x(k)$ (iii) $y(n) = ax(n)$

(iv) $y(n) = x(n)\,e^n$ (v) $y(n) = a^{x(n)}$

Solution :

(i) Given $y(n) = \cos x(n)$

For the system to be stable, it has to satisfy the condition.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

If $x(n) = \delta(n)$, then the impulse response $h(n) = \cos \delta(n)$.

$$\boxed{\begin{aligned} \delta(n) &= 1 \text{ for } n = 0 \\ &= 0 \text{ for } n \neq 0 \end{aligned}}$$

For $n = 0$; $h(0) = \cos 1 = 0.54$

For $n = 1$; $h(1) = \cos 0 = 1$

For $n = 2$; $h(2) = \cos 0 = 1$

For $n = -1$; $h(-1) = \cos 0 = 1$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \ldots |h(-2)| + |h(-1)| + |h(0)|$$

$$+ |h(1)| + 1\ h(2) | + \ldots |h(\infty)|$$

$$= 1 + 1 + \ldots 1 + 1 + 0.54 + 1 + 1 + \ldots 1$$

$$= \infty$$

The system is unstable.

(ii) $y(n) = \sum_{k=-\infty}^{n+1} x(k)$

$x(k) \rightarrow \delta(k)$

$y(n) \rightarrow h(n)$

For the system to be stable $\sum_{k=-\infty}^{\infty} |h(n)| < \infty$

For the given system

$$h(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

$$h(n) = \sum_{k=-\infty}^{n+1} \delta(k)$$

For $n = -2$

$$h(-2) = \sum_{k=-\infty}^{-1} \delta(k) = 0$$

$$\boxed{\begin{aligned} \delta(k) &= 0 \text{ for } k \neq 0 \\ &= 1 \text{ for } k = 0 \end{aligned}}$$

For $n = -1$

$$h(-1) = \sum_{k=-\infty}^{0} \delta(k) = 1$$

For $n = 1$

$$h(0) = \sum_{k=-\infty}^{0} \delta(k) = 1$$

For $n = 1$

$$h(1) = \sum_{k=-\infty}^{2} \delta(k) = 1$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-1}^{\infty} |h(n)| = 1 + 1 + 1 \ldots \infty = \infty$$

The condition is not satisfied, therefore, the system is unstable.

(iii) $y(n) = ax(n)$

The impulse response is given by

$$\boxed{\begin{aligned}\delta(n) &= 0 \text{ for } n \neq 0 \\ &= 1 \text{ for } n = 0\end{aligned}}$$

$$h(n) = a\,\delta(n)$$

For $n = -1$

$$h(-1) = a\,\delta(-1) = 0$$

For $n = 0$

$$h(0) = a\,\delta(0) = a$$

For $n = 1$

$$h(1) = a\,\delta(1) = 0$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \ldots |h(-1)| + |h(0)| + |h(1)| + \ldots |h(\infty)|$$

$$= 0 + 0 + \ldots 0 + a + 0 \ldots 0$$

$$= a$$

The system is stable, if $|a| < \infty$

(iv) $y(n) = x(n)\,e^{n}$

The impulse response is given by

$$h(n) = \delta(n)\,e^{n}$$

For $n = -1$

$$h(-1) = \delta(-1)\,e^{-1}$$

$$= 0$$

For $n = 0$

$$h(0) = \delta(0)e^{0} = 1$$

For $n = 1$

$$h(1) = \delta(1)\,e^{1} = 0$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(\infty)| + \ldots |h(-1)| + |h(0)| + |h(1)| + \ldots |h(\infty)|$$

$= 0 + \ldots + 0 + 1 + 0 + \ldots 0$

$= 1 < \infty$

Therefore, the system is stable.

(v) $y(n) = a^{x(n)}$

The impulse response is given by

$h(n) = a^{\delta(n)}$

For $n = 0$; $h(0) = a^{\delta(0)}$ = a

For $n = -1$ $h(-1) = a^0 = 1$

For $n = 1$; $h(1) = a^{\delta(0)} = 1$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(\infty)| + \ldots |h(0)| + |h(1)| + |h(2)| + \ldots |h(\infty)|$$

$$= 1 + \ldots a + 1 + 1 + \ldots 1$$

$$= \infty$$

The system is unstable.

Example 2.39

Find the discrete convolution of the following sequences

(a) $x(n) = \{\underset{\uparrow}{1}, 2, -1, 1\}$ $h(n) = \{1, \underset{\uparrow}{0}, 1, 1\}$ (b) $u(n) * u(n-3)$

(c) $2^n u(-n+2) * u(n-3)$ (d) $\cos\left(\dfrac{\pi n}{2}\right) u(n) * u(n-1)$

(e) $x(n) = e^{-n^2}$; $h(n) = 3n^2$

Solution

(a) The starting value of $n = n_1 + n_2$

$= 0 + (-1) = -1$

h(n) \ x(n)	1	2	−1	1
1	1	2	−1	1
0	0	0	0	0
1	1	2	−1	1
1	1	2	−1	1

$y(n) = \{1, \underset{\uparrow}{2}, 0, 4, 1, 0, 1\}$

(b) Let $x(n) = u(n)$ and $h(n) = u(n-3)$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

We have

$x(k) = 0$ for $k < 0$ and

$h(n-k) = 0$ for $k > n-3$

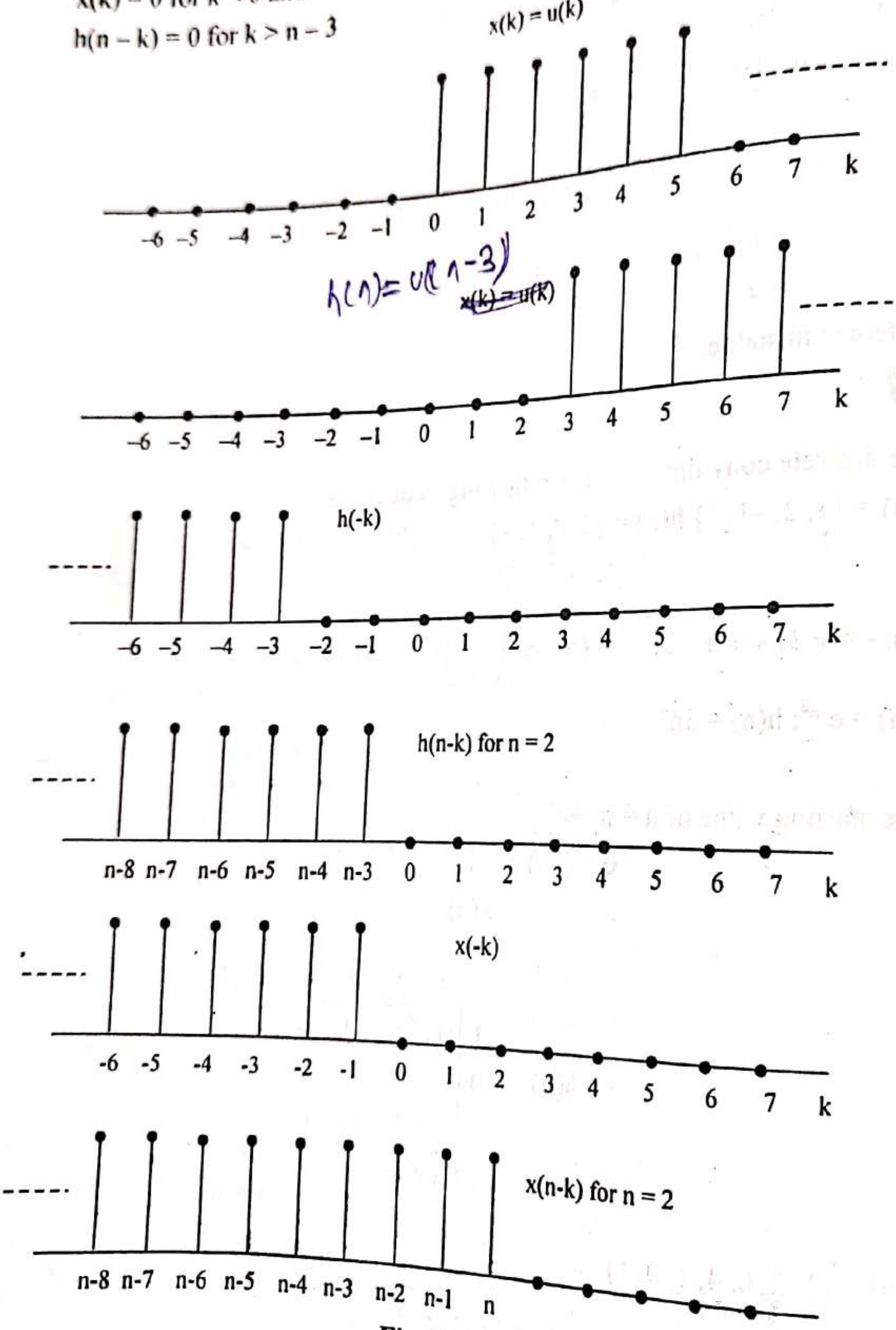

Fig. 2.35

Therefore

$$y(n) = \sum_{k=0}^{n-3} x(k)\, h(n-k) = \sum_{k=0}^{n-3} 1 = n-3+1 = \boxed{n-2}$$

$$= \sum_{k=0}^{n-3} 1 \qquad \boxed{\sum_{k=n_1}^{n_2} 1 = n_2 - n_1 + 1} \qquad \sum_{n=0}^{N} 1 = \boxed{N+1}$$

$$= n-3-0+1 = n-2$$

(or)

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

we have

$$h(k) = 0 \text{ for } k < 3$$
$$x(n-k) = 0 \text{ for } k > n$$

$$y(n) = \sum_{k=3}^{n} h(k)\, x(n-k)$$

$$= \sum_{k=3}^{n} 1$$

$$= n-3+1 = n-2$$

(c) $2^n u(-n+2) * u(n-3)$

Let $x(n) = 2^n u(-n+2)$ and

$$h(n) = u(n-3)$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

For $-\infty \le n \le 5$

$$\boxed{\begin{array}{l} h(n-k) = 0 \\ \text{for } k > n-3 \end{array}}$$

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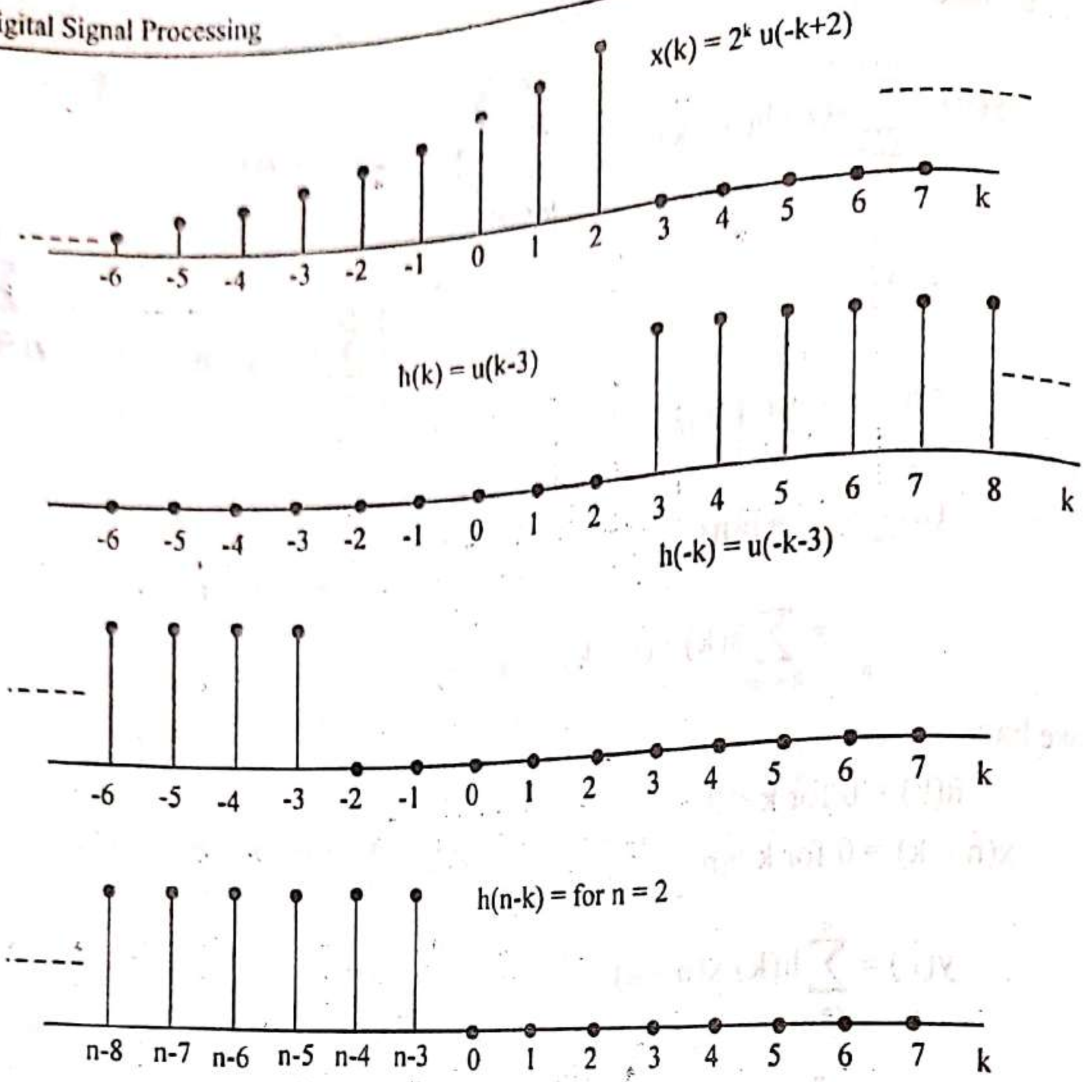


Fig. 2.36

$$y(n) = \sum_{k=-\infty}^{n-3} 2^k$$

$$= \left[2^{n-3} + 2^{n-4} + \ldots\right]$$

$$= 2^{n-3}\left[1 + \frac{1}{2} + \ldots\right]$$

$$= 2^{n-3} \cdot \frac{1}{1-\frac{1}{2}} = 2^{n-2}$$

For n > 5

$$y(n) = \sum_{k=-\infty}^{2} 2^k = \left[2^2 + 2 + 1 + \frac{1}{2} \ldots\right] = \frac{4}{1-\frac{1}{2}} = 8$$

$$= \left[4 + 2 + 1 + \frac{1}{2} + \ldots\right]$$

(d) $x(n) = \cos\left(\frac{\pi n}{2}\right)u(n)$

$h(n) = u(n-1)$

The above sequences can be represented as

$x(n) = \{\underset{\uparrow}{1}, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, ...\}$

$h(n) = \{\underset{\uparrow}{0}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1...\}$

h(n) \ x(n)	1	0	-1	0	1	0	-1	0	1	0	-1	...
0	0	0	0	0	0	0	0	0	0	0	0	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
1	1	0	-1	0	1	0	-1	0	1	0	-1	...
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.

∴ $y(n) = \{0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0...\}$

(or)

$$y(n) = x(n) * h(n)$$

$$= \sum_{n=-\infty}^{\infty} x(k)\, h(n-k)$$

Given $x(k) = \cos\frac{\pi k}{2}u(k)$ and $h(n-k) = u(n-k-1)$

$x(k) = 0$ for $k < 0$

$h(n-k) = 0$ for $k > n-1$

Therefore

$$y(n) = \sum_{k=0}^{n-1}\cos\frac{\pi k}{2}$$

$$= \text{Real part of}\left[\sum_{k=0}^{n-1}e^{j\pi k/2}\right]$$

$$= \text{Re}\left[1 + e^{j\pi/2} + e^{j\pi} + \ldots n \text{ terms}\right]$$

$$= \text{Re}\left[\frac{e^{j\pi/2}-1}{e^{j\pi/2}-1}\right] = \text{Re}\left[\frac{e^{j\pi/2}-1}{-1+j}\right]$$

$$= \text{Re}\left[\frac{(e^{j\pi/2}-1)(-1-j)}{2}\right] = \text{Re}\left[\frac{-e^{j\pi/2}+1-j^{j\pi/2}+j}{2}\right]$$

$$= \frac{1}{2}\text{Re}\left[-\cos\frac{\pi n}{2} - j\sin\frac{\pi n}{2} + 1 - j\cos\frac{\pi n}{2} + \sin\frac{\pi n}{2} + j\right]$$

$$= \frac{1}{2}\left[1 - \cos\frac{\pi n}{2} + \sin\frac{\pi n}{2}\right]$$

(e) Given

$x(n) = e^{-n^2}$; $h(n) = 3n^2$

$$y(n) = \sum_{k=-\infty}^{\infty}x(k)h(n-k) = \sum_{k=-\infty}^{\infty}e^{-k^2}3(n-k)^2$$

$$= 3\left\{\sum_{k=-\infty}^{\infty}e^{-k^2}n^2 + \sum_{k=-\infty}^{\infty}e^{-k^2}(-2nk) + \sum_{k=-\infty}^{\infty}e^{-k^2}k^2\right\}$$

$$= 3\left\{\sum_{k=-\infty}^{\infty}e^{-k^2}n^2 + \sum_{k=-\infty}^{\infty}e^{k^2}(-2nk) + \sum_{k=-\infty}^{\infty}e^{-k^2}k^2\right\}$$

$$=3\left\{n^2\sum_{k=-\infty}^{\infty}e^{-k^2}-2n\sum_{k=-\infty}^{\infty}e^{k^2}k+\sum_{k=-\infty}^{\infty}k^2e^{-k^2}\right\}$$

$$\sum_{k=-\infty}^{\infty}e^{-k^2}=\ldots e^{-4}+e^{-1}+1+e^{-1}+e^{-4}+e^{-9}+\ldots$$

$$=1+2\left(e^{-1}+e^{-4}+e^{-9}+\ldots\right)$$

$$=1+2\,(0.3863)$$

$$=1.7726$$

$$\sum_{k=-\infty}^{\infty}e^{-k^2}k=\ldots-3e^{-9}-2e^{-4}-1e^{-1}+0+1e^{-4}+2e^{-4}+3e^{-9}+\ldots$$

$$=0$$

$$\sum_{k=-\infty}^{\infty}k^2e^{-k^2}=\ldots16e^{-16}-9e^{-9}-4e^{-1}+e^{-1}+0+e^{-1}+4e^{-4}+9e^{-9}+16e^{-16}+\ldots$$

$$=2\left\{e^{-1}+4e^{-4}+9e^{-9}+16e^{-16}+\ldots\right\}$$

$$=0.8845$$

$$y(n)=3\left\{1.7726n^2+0+0.8845\right\}=5.318n^2+2.654$$

Example 2.40

Determine the stability of the system

$\boxed{\text{Imp}}$

$$y(n)-\frac{5}{2}y(n-1)+y(n-2)=x(n)-x(n-1)$$

If the difference eqn given then find h(n) through homogeneous eq n

Solution :

For the system to be stable

$$\sum_{n=-\infty}^{\infty}|\,h(n)\,|<\infty$$

Substituting $x(n)=0$ and $y(n)=\lambda^n$ in the difference equation we get

$$\lambda^n-\frac{5}{2}\lambda^{n-1}+\lambda^{n-2}=0$$

$$\lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

$$\lambda_1 = 2;\ \lambda_2 = \frac{1}{2}$$

$$y(n) = C_1(2)^n + C_2\left(\frac{1}{2}\right)^n \qquad \text{... (I)}$$

For n = 0

$$y(0) = C_1 + C_2$$

For n = 1

$$y(1) = 2C_1 + \frac{1}{2}C_2 \qquad \text{... (II)}$$

From the difference equation we find , $x(n) = \delta(n)$
$\delta(0) = 1$

$$y(0) = 1$$

$$y(1) = \frac{3}{2} \qquad \text{... (III)}$$

comparing Eq. (I), Eq. (II) and Eq.(III) we have

$$C_1 + C_2 = 1$$

$$2C_1 = \frac{1}{2}C_2 = \frac{3}{2}$$

Solving for C_1 and C_2 we obtain $C_1 = \frac{2}{3}$ and $C_2 = \frac{1}{3}$

$$h(n) = \frac{2}{3}(2)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n \text{ for } n \geq 0$$

$$= \boxed{\left[\frac{2}{3}(2)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n\right]u(n)}$$

For the system to be stable

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} \left| \underbrace{\frac{2}{3}(2)^n}_{\text{diverges}} + \underbrace{\frac{1}{3}\left(\frac{1}{2}\right)^n}_{\text{converges}} \right| = \infty$$

Therefore, the system is unstable.

SUMMARY

1. Systems are broadly classified as continuous-time systems and discrete-time system. Continuous-time systems deal with continuous-time signals and discrete-time systems deal with discrete-time signals.
2. Both continuous-time and discrete-time systems have several basic properties. Out of these several basic properties of systems, two properties namely linearity and time-invariance play a vital role in the analysis of signals and systems. If a system has both the linearity and time-invariance properties, then this system is called **Linear-time Invariant** System.
3. We study linear-time invariant systems because of the fact that most of the practical and physical processes around us can be modelled in the form of linear-time invariant systems.
4. Linear-time invariant systems may be analyzed in detail very easily and thus providing some fundamental aspects for the complex analysis of signals and systems.
5. Both continuous-time and discrete-time, linear-time-invariant (LTI) systems exhibit one important characteristics that the superposition theorem can be applied to find the response $y(t)$ to a given input $x(t)$.
6. To find the response of a LTI system to any given function first we have to find the response of LTI system to an unit impulse called unit **impulse response** of LTI system.
7. The impulse response of a continuous-time or discrete-time LTI system is the output of the system due to an unit impulse input applied at time $t = 0$ or $n = 0$. Here, $\delta(t)$ is the unit impulse input in continuous-time and $h(t)$ is the unit-impulse response of continuous-time LTI system. In other words, continuous-time unit-impulse response $h(t)$ is the output of a continuous-time system when applied input $x(t)$ is equal to unit impulse function $\delta(t)$.
8. For a discrete-time system, discrete time impulse response $h(n)$ is the output of a discrete-time system when applied input $x(n)$ is equal to discrete-time unit impulse function $\delta(n)$. Here, $\delta(n)$ is the unit-impulse input in discrete-time and $h(n)$ is the unit-impulse response of discrete-time LTI system.
9. Therefore, any LTI system can be completely characterized in terms of its unit **impulse response.**

10. The discrete-time output signal y(n) of this system may be expressed as

$$y(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k)$$

The above expression for discrete-time output signal y(n) is called the convolution sum as against the convolution integral for continuous-time LTI system.

11. The LTI systems have a number of properties not exhibited by other systems. These are as under:

(i) Commutative property of LTI systems.

(ii) Distributive property of LTI systems

(iii) Associative property of LTI systems.

(iv) Static and dynamic LTI systems

(v) Invariability of LTI systems

(vi) Causality of LTI systems

(vii) Stability of LTI systems

(viii) Unit-step response of LTI systems

13. According to commutative property, for a discrete-time system.

The output $$y(n) = x(n)*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

or $$y(n) = h(n)*x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

15. For discrete-time LTI system, the distributive property is expressed as

The output $$y(n) = x(n)*\{h_1(n) + h_2(n)\}$$

$$y(n) = x(n)*h_1(n) + x(n)*h_2(n)$$

17. Static systems are also known as memoryless systems. A system is known as static if its output at any time depends only on the value of the input at the same time.

18. A system is known as invertible only if an inverse system exists which, when cascaded (connected in series) with the original system, produces an output equal to the input at first system. If an LTI system is invertible then it will have a LTI inverse system.

QUESTIONS AND ANSWERS

Q.1 ***What do you understand by the terms: signal and signal processing.***

Ans A signal is defined as any physical quantity that varies with time, space, or any other independent variable.

Signal processing is any operation that changes the characteristics of a signal. These characteristics include the amplitude, shape, phase and frequency content of a signal.

Q.2 ***What is Deterministic signal? Give example.***

Ans A Deterministic signal is a signal exhibiting no uncertainty of value at any given instant of time. Its instantaneous value can be accurately predicted by specifying a formula, algorithm or simply its describing statement in words.

Example: $v(t) = A_0 \sin \omega t$

Q.3 ***What is random signal ?***

Ans A random signal is a signal characterized by uncertainty before its actual occurrence.

Example: Noise

Q.4 ***Define (a) Periodic signal (b) Non-periodic signal.***

Ans A signal $x(n)$ is periodic with period N if and only if $x(n + N) = x(n)$ for all n.

If there is no value of N that satisfies the above equation the signal is called nonperiodic or aperiodic.

Q.5 ***Define the following***

(a) Analog signal (b) Discrete-time signal (c) Digital signal

Ans (a) An analog signal is a function having an amplitude varying continuously for all values of time. Hence, an analog signal is continuous in both time and amplitude.

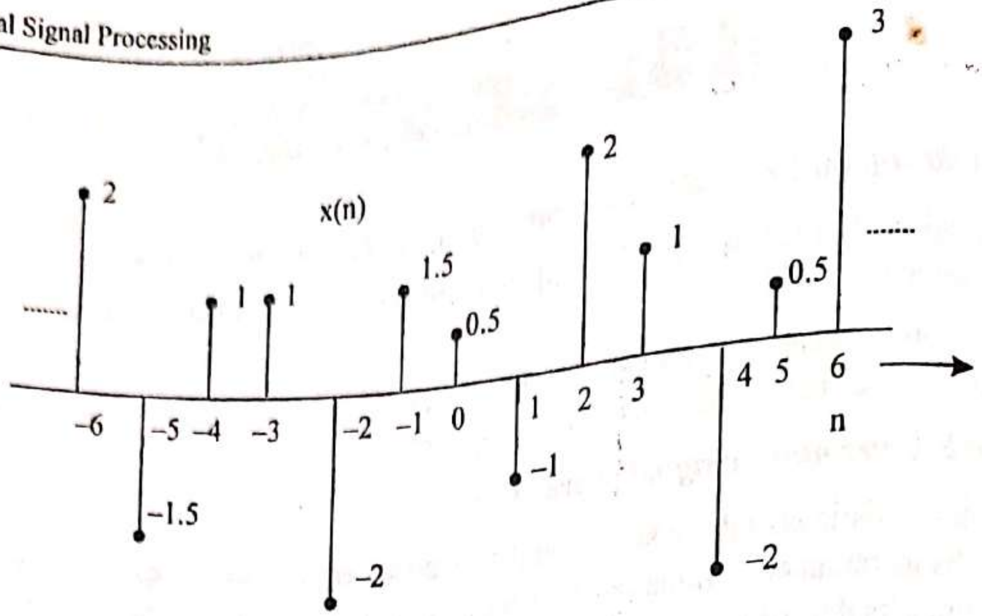
Examples of analog signals are the sinusoidal function, the step function, output from a microphone.

(b) A discrete-time signal is a function defined only at particular time instants. It is discrete in time but continuous in amplitude. An example is temperature recorded at regular intervals of time in a day.

(c) A digital signal is a special form of discrete-time signal which is discrete in both time and amplitude, obtained by quantizing each value of the discrete-time signal. These signals are called digital because their samples are represented by numbers or digits. Examples of digital signals include the dot-dash Morse code, the output from a digital computer etc.

Q.6 ***Give the analytical and graphical representation of an arbitrary sequence.***

Ans Graphical representation of an arbitrary sequence is given by



We can write any arbitrary sequence x(n) into a sum of unit sample sequence. If we multiply two sequences x(n) and delayed unit impulse $\delta(n-k)$, the result is another sequence that is zero everywhere except at $n = k$, where its value is x(k). Thus

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

If we repeat this multiplication over all possible delays, $-\infty < k < \infty$, and sum all the product sequences, the result will be a sequence equal to the sequence x(n), that is

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Q.7 ***What are the different types of operations performed on discrete-time signals?***

Ans The different types of operations performed on discrete-time signals are

(1) Delay of a signal (2) Advance of a signal (3) Folding or reflection of a signal (4) Time scaling (5) Amplitude scaling (6) Addition of signals (7) Multiplication of signals.

Q.8 ***What is the property of shift-invariant system?***

(or)

What is a time-invariant system?

(or)

What is a shift-invariant system? Give an example.

Ans If the input-output relation of a system does not vary with time, the system is said to be time-invariant or shift-invariant.

If the output signal of a system shifts k units of time upon delaying the input signal by k units, the system under consideration is a time-invariant system.

Example: $y(n) = x(n) + x(n-1)$

Q.9 *What is a causal system? Give an example.*

(or)

What is a causal system?

Ans A system is said to be causal if the output of the system at any time n depends only on present and past input, but does not depend on future inputs.

This can be represented mathematically as

$$y(n) = F[x(n), x(n-1), x(n-2)\ldots]$$

Example: $y(n) = x(n) + x(n-1)$

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

Q.10 *What is an LTI system?*

Ans An LTI system is one which possess two of the basic properties linearity and time-invariance.

Linearity: An LTI system obeys superposition principle which states that the output of the system to a weight sum of inputs is equal to the corresponding weighted sum of the outputs to each of the individual inputs.

Time invariance: If the input-output relation of a system does not vary with time, the system is said to be time-invariant.

Q.11 ***Define unit sample response (impulse response) of a system and what is its significance.***

Ans The response or output signal designated as h(n), obtained from a discrete-time system when the input signal is a unit sample sequence (unit impulse), is known as the unit sample response (impulse response).

The output y(n) of an LTI system for an input signal x(n) can be obtained by convolving the impulse response h(n) and the input signal x(n).

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Q.12 ***What is causality condition for an LTI system?***

Ans The necessary and sufficient condition for causality of an LTI system is, its unit sample response h(n) = 0 for negative values of n i.e.

$$h(n) = 0 \quad \text{for} \quad n < 0$$

Q.13 ***What is condition for system stability?***

(or)

What is the necessary and sufficient condition on the impulse response for stability?

Ans The necessary and sufficient condition guaranteeing the stability of a linear time-invariant system is that its impulse response is absolutely summable

$$\text{i.e., } \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Q.14 ***What do you understand by linear convolution?***

(or)

What is meant by discrete convolution?

Ans The convolution of discrete-time signals is known as discrete convolution. Let x(n) be the input to an LTI system and y(n) be the output of the system. Let h(n) be the response of the system to an impulse. The output y(n) can be obtained by convolving the impulse response h(n) and the input signal x(n)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \text{ (or) } y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

The above equation that gives the response y(n) of an LTI system as a function of the input signal x(n) and the impulse response h(n) is called a convolution sum.

Q.15 ***What are FIR and IIR systems?***

Ans FIR system: This type of system has an impulse response which is zero outside a finite time interval.

Example: $h(n) = 0$, for $n < 0$ and $n > N$

IIR system: An IIR system exhibits an impulse response of infinite duration.

Q.16 ***What is the property of recursive and non recursive systems?***

Ans Recursive system: This type of system has the property that output y(n) at time n is a function of any number of past outputs

$y(n-1), y(n-2), \ldots y(n-N)$ as well as present and past inputs

$x(n), x(n-1), x(n-2) \ldots x(n-N)$.

i.e., $y(n) = T[x(n), x(n-1), \ldots x(n-N), y(n-1), y(n-2) \ldots y(n-N)]$

Non recursive system: In this kind of system, the output y(n) depends only on the present and past input signal values, i.e.,

$$y(n) = T[x(n), x(n-1), x(n-2), \ldots x(n-N)]$$

Q.17 ***A causal system is one whose impulse response $h(n) = 0$ for $n < 0$. True/False***

Ans True

Q.18 ***A recursive system described by a linear constant difference equation is linear and time-in variant. True/False***

Ans True

Q.19 ***A linear system is stable if its impulse response is absolutely summable, True/False***

Ans True

Q.20 ***How you can find step response of a system if the impulse response h(n) is known ?***

Ans We have

$$y(n) = x(n) * h(n)$$

For input $x(n) = u(n)$

$$y(n) = u(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} u(n-k)h(k)$$

$$= \sum_{k=-\infty}^{n} h(k) \qquad \boxed{\because y(n-k) = 0 \text{ for } k > n}$$

Q.21 ***Determine the unit step response of the LTI system with impulse response***

$h(n) = a^n u(n) \mid a \mid < 1.$ Imp

Ans Unit step response

$$y(n) = \sum_{k=-\infty}^{n} h(k)$$

$$= \sum_{k=0}^{n} a^k$$

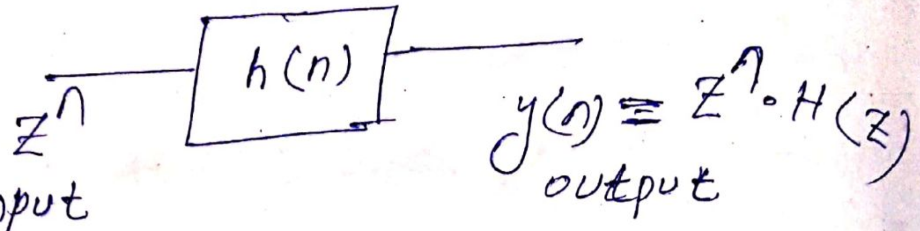
$$= \frac{1-a^{n+1}}{1-a}$$

Q.22 ***Define Fourier transform of a sequence.***

Ans The Fourier transform of a finite energy discrete-time signal x(n) is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

DSP (Z-Transform) 14-2-2020



$$H(Z) = Z[h(n)] = \sum_{n=-\infty}^{\infty} h(n) \cdot Z^{-n}$$

If input is $x(n)$ Then it's Z-Transform is $Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$

ROC :- Region of convergence of $X(Z)$ is set of all the values of Z for which $X(Z)$ has finite value.

$Z = r \cdot e^{jw}$, r = Radius of the circle in Z-domain

↳ If $x(n)$ is causal signal i.e. $x(n) = 0$ for $n < 0$, Then The Z-Transform is

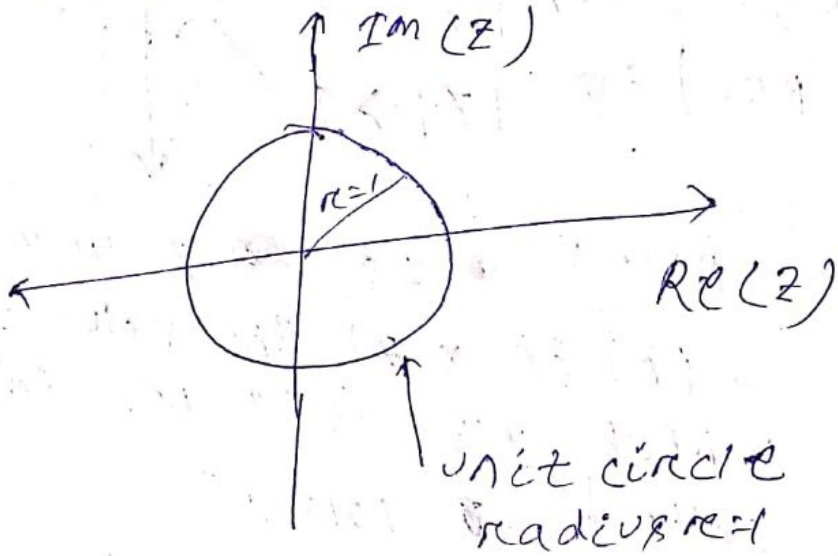
$$X(Z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

Here it is one sided Z-Transform. It contains negative powers (-ve) of Z in $X(Z)$ expression.

→ If $x(n)$ is noncausal Discrete Time signal i.e. $x(n)=0$ for $n>0$ Then its Z-Transform is

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{-1} x(n) \cdot Z^{-n}$$

It contains positive powers(+ve) of Z in above expression. It is also one sided Z-Transform.



→ If $x(n) = u(n)$.

$$X(Z) = \sum_{n=0}^{\infty} u(n) \cdot Z^{-n}$$

$$= u(0) \cdot Z^{-0} + u(1) \cdot Z^{-1} + u(2) \cdot Z^{-2} + u(3) \cdot Z^{-3} + \cdots$$

$$= 1 \cdot 1 + 1 \cdot Z^{-1} + 1 \cdot Z^{-2} + 1 \cdot Z^{-3} + \cdots$$

$$= 1 + (Z^{-1})^1 + (Z^{-1})^2 + (Z^{-1})^3 + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots \quad [x = Z^{-1}]$$

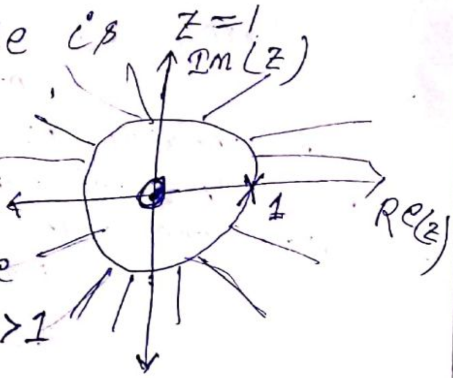
$$= \frac{1}{1-x} = \boxed{\frac{1}{1-Z^{-1}}} = \frac{1}{1-\frac{1}{Z}} = \frac{1}{\frac{Z-1}{Z}}$$

$$= \boxed{Z/Z-1}$$

$Z[u(n)] = \frac{z}{z-1}$

Here zero is $z = 0$

Here pole is $z = 1$



Here region of convergence (ROC) is $|z| > 1$

↳ **pole:-** If Z-Transform of $x(n)$ is $X(z)$. The value of z for which $X(z)$ will be infinite is called pole.

↳ **zero:-** The value of z for which $X(z)$ will be zero (0) is called zero.

$x(n) = \{1, \underset{\uparrow n=0}{2}, 3\}$ find $X(z)$.

$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} = \sum_{n=-1}^{1} x(n) \cdot z^{-n}$

$= x(-1) \cdot z^{-(-1)} + x(0) \cdot z^{-0} + x(1) \cdot z^{-1}$

$= 1 \cdot z^{1} + 2 \cdot z + 3 \cdot z^{-1}$

Here ROC is all values of z except $z = 0$ and $z = \infty$.

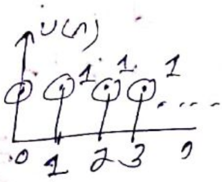
0 → ZERO

X → POLE

① **causal infinite duration signal :-**

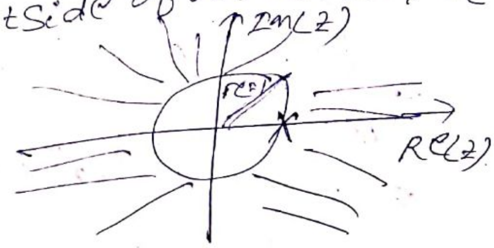
EX:- $u(n) = 1$; $n \geq 0$

$= 0$; for $n < 0$



$Z[u(n)] = \frac{z}{z-1}$

ROC:- outside of outeremost pole



① $Z[\delta(n)] = 1$

② If $Z[x(n)] = X(z)$ then

$Z[x(n-k)] = z^{-k} \cdot X(z)$, ROC: Entire z-plane except $z=0$

$Z[x(n+k)] = z^{k} \cdot X(z)$

③ $Z[\delta(n-k)] = z^{-k} \cdot Z[\delta(n)]$

$= z^{-k} \cdot 1 = z^{-k}$

④ $Z[\delta(n+k)] = z^{k} \cdot Z[\delta(n)] = z^{k} \cdot 1 = z^{k}$

ROC:- Entire z-plane except $z = \infty$.

Que:- Determine the Z-Transform and ROC of the signal $x(n) = a^n \cdot u(n)$

Sol^n:- The given signal is causal and infinite duration.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot \underbrace{u(n)}_{1} \cdot Z^{-1}$$

$u(n) = 1;$ for $n \geq 0$

$$= \sum_{n=0}^{\infty} a^n \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} (a Z^{-1})^n \qquad \text{put } az^{-1} = x$$

$$= x^0 + x^1 + x^2 + x^3 + x^4 + \ldots$$

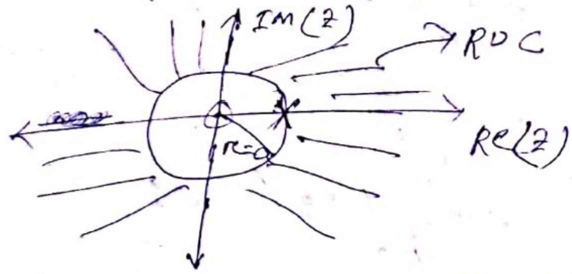
$$= \frac{1}{1-x} = \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{1}{\frac{z-a}{z}}$$

$$= \boxed{\frac{Z}{Z-a}}$$

for infinite duration causal signal, the ROC is outside of outermost pole.

Here one pole $z = a$



ZERO at $Z = 0$, POLE at $Z = a$

ROC:- $|Z| > a$

Que:- Find the Z-Transform and the ROC of the signal

$$x(n) = -b^n \cdot u(-n-1)$$

Sol^n:- $u(-n-1) = 1$, for $n \leq -1$

$= 0$, for $n \geq -1$

put $n = -1$, $\Rightarrow u[-(-1)-1] = u[1-1] = u[0] = 1$

put $n = -2$, $\Rightarrow u[-(-2)-1] = u[2-1] = u(1) = 1$

$$X(Z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -b^n \cdot \underbrace{u(-n-1)}_{1} \cdot Z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n \cdot Z^{-n} = -\sum_{n=1}^{\infty} \frac{1}{b^n \cdot Z^{-n}}$$

$$= -\sum_{n=1}^{\infty} b^{-n} \cdot z^{n} = -\sum_{n=1}^{\infty} (b^{-1}z)^{n}$$

$$= -\left[\sum_{n=1}^{\infty} (b^{-1}z)^{n} + (b^{-1}z)^{0} - (b^{-1}z)^{0}\right]$$

$$= -\left[\sum_{n=0}^{\infty} \underbrace{(b^{-1}z)}_{x}{}^{n} - 1\right]$$

$$= -\left[\frac{1}{1-x} - 1\right]$$

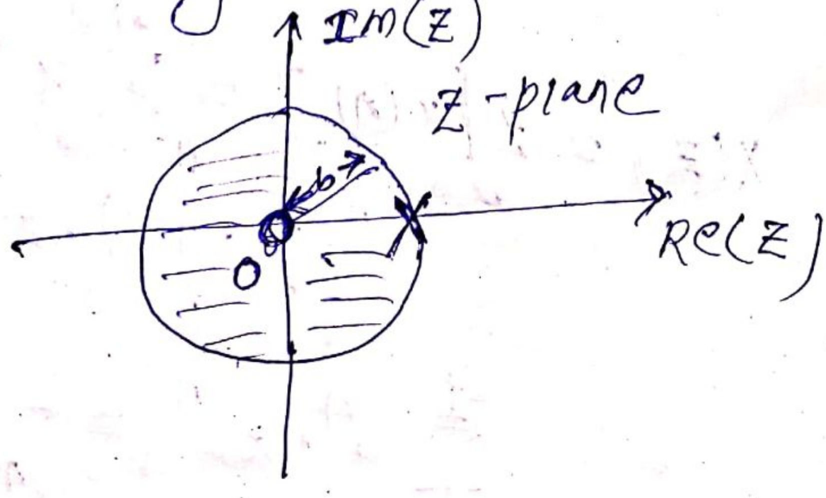
$$= -1\left[\frac{1}{1-b^{-1}z} - 1\right] = -\left[\frac{1}{1-\frac{z}{b}} - 1\right]$$

$$= -\left[\frac{1}{\frac{b-z}{b}} - 1\right] = -\left[\frac{b}{b-z} - 1\right]$$

$$= 1 - \frac{b}{b-z} = \frac{b-z-b}{b-z} = \frac{-z}{b-z}$$

$$ROC :- b-z > 1 \Rightarrow b > z \Rightarrow \boxed{z < b}$$

The ROC is now the interciore of a circle having radius b



18-2-202

Digital signal processing

Z-TRANSFORM :-

$$Z[a^n \cdot u(n)] = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \; ROC: |Z| > a$$

$$Z[-a^n \cdot u(-n-1)] = \frac{1}{1-az^{-1}}, \; ROC: |Z| < a$$

$$= \frac{z}{z-a}$$

ROC of Two sided sequence :-

↳ The ROC of a causal signal is exterior of a circle of radius r. The ROC of of a anticausal signal is interior of a circle of radius r.

Let us consider a Two sided sequence is

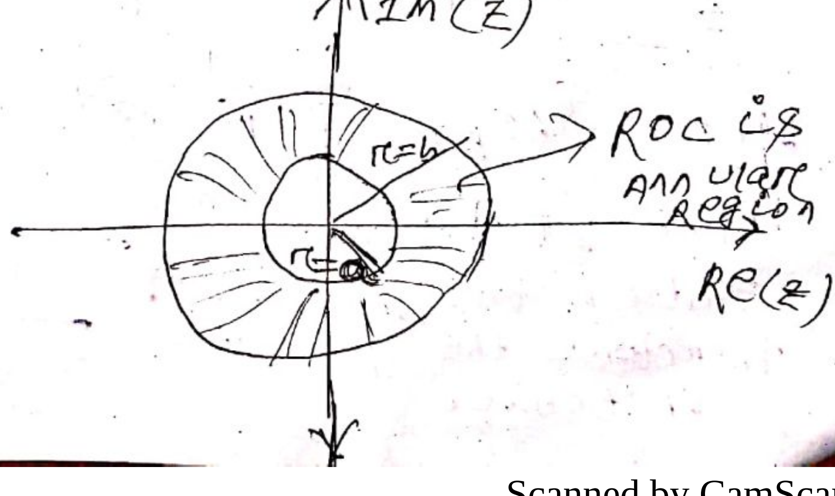
$$x(n) = a^n \cdot u(n) + b^n \cdot u(-n-1)$$

$$X(Z) = \frac{1}{1-aZ^{-1}} - \frac{1}{1-bZ^{-1}}$$

$\downarrow$ ROC: $|Z| > a$ $\downarrow$ ROC: $|Z| < b$

So combined ROC is

$$a < |Z| < b$$



Stability and ROC:-

$y(n) = x(n) * h(n)$ $\quad x(n) \rightarrow \boxed{\text{system } h(n)} \rightarrow y(n)$

Let $h(n)$ is the impulse Response of a causal (or) Non causal Linear Time invariant system and $H(Z)$ be the Z-Transform of $h(n)$. Then stability of the system can be found from ROC using the following Theorem.

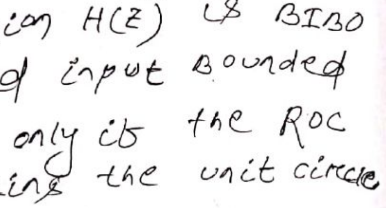
Theorem:- An LTI system with the system function $H(Z)$ is BIBO stable (Bounded input Bounded output) if and only if the ROC for $H(Z)$ contains the unit circle.

Que:- Find the stability of the system whose impulse Response is $h(n) = (2)^n . u(n)$

Sol:- $H(Z) = Z[(2)^n . u(n)]$

$= \frac{1}{1-2 \cdot Z^{-1}} = \frac{Z}{Z-2}$; ROC is $|Z| > 2$

Here, the ROC is $|Z| > 2$. It does not contain the unit circle. Therefore the system is unstable.



Properties of Z-Transform:-

(1) Linearity:- If $Z[x_1(n)] = X_1(Z)$; ROC $= R_1$

$Z[x_2(n)] = X_2(Z)$; ROC $= R_2$

Then $x(n) = a \cdot x_1(n) + b \cdot x_2(n)$ having Z-Transform is

$Z[x(n)] = X(Z) = a \cdot Z[x_1(n)] + b \cdot Z[x_2(n)]$

$= a \cdot X_1(Z) + b \cdot X_2(Z)$

ROC is : $R_1 \cap R_2$

a, b are constants

Que:- The signal is given by $x(n) = [2(3^n) - 3(4^n)]u(n)$. Determine Z-Transform using Linearity property.

Sol:- $x(n) = 2(3)^n . u(n) - 3(4)^n . u(n)$

$Z[x(n)] = X(Z) = 2 Z[(3)^n u(n)] - 3 \cdot Z[(4)^n u(n)]$

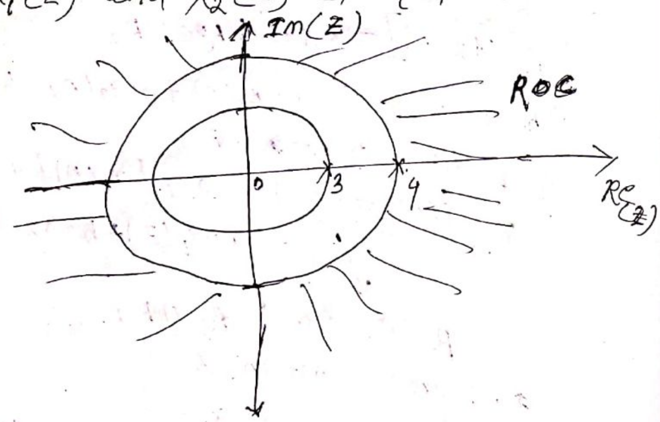
$= 2 \cdot \frac{1}{1-3 \cdot Z^{-1}} - 3 \cdot \frac{1}{1-4 \cdot Z^{-1}}$

$= 2 \cdot \frac{Z}{Z-3} - 3 \cdot \frac{Z}{Z-4}$

ROC(R_1) : $|Z| > 3$ $\quad$ ROC(R_2) : $|Z| > 4$

$R_1 \cap R_2$ is ROC $|Z| > 4$

The intersection of ROC of $X_1(Z)$ and $X_2(Z)$ is $|Z| > 4$.



---x---

Que: Consider the Z-Transform

$$X(Z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})},$$

ROC: $|Z| > \frac{1}{2}$, $|Z| > \frac{1}{4}$.

Determine $x(n)$.

Sol^n:- $$X(Z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}}$$

$$A = \lim_{z^{-1}\to 2} (1-\tfrac{1}{2}z^{-1}) \cdot X(Z)$$

$$= \lim_{z^{-1}\to 2} \cancel{(1-\tfrac{1}{2}z^{-1})} \cdot \frac{1}{\cancel{(1-\frac{1}{2}z^{-1})}(1-\frac{1}{4}z^{-1})}$$

$$= \frac{1}{1-\frac{1}{4}\times 2} = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2}$$

$$= \boxed{2}$$

$$B = \lim_{z^{-1}\to 4} (1-\tfrac{1}{4}z^{-1}) \cdot X(Z)$$

$$= \lim_{z^{-1}\to 4} \cancel{(1-\tfrac{1}{4}z^{-1})} \cdot \frac{1}{(1-\frac{1}{2}z^{-1})\cancel{(1-\frac{1}{4}z^{-1})}}$$

$$= \lim_{z^{-1}\to 4} \frac{1}{1-\frac{1}{2}\cdot z^{-1}} = \frac{1}{1-\frac{1}{2}\times 4} = \frac{1}{1-2}$$

$$= \frac{1}{(-1)} = \boxed{-1}$$

$$X(Z) = 2 \cdot \frac{1}{1-\frac{1}{2}z^{-1}} - 1 \cdot \frac{1}{1-\frac{1}{4}\cdot z^{-1}}$$

$$Z^{-1}[X(Z)] = x(n)$$
$$= 2 \cdot Z^{-1}\left[\frac{1}{1-\frac{1}{2}\cdot z^{-1}}\right] - 1 \cdot Z^{-1}\left[\frac{1}{1-\frac{1}{4}z^{-1}}\right]$$

$$= 2 \cdot \left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n \cdot u(n)$$

$$Z^{-1}[a^n \cdot u(n)] = Z^{-1}[-a^n \cdot u(-n-1)]$$
$$= \frac{1}{1-a\cdot z^{-1}}$$

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$Z[x(n-K)] = Z^{-K} \cdot X(Z)$

proof:-

$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$

put $n-K = l$

$Z[x(n-K)] = Z[x(l)] = \sum_{l=-\infty}^{\infty} x(l) \cdot Z^{-l}$

$= \sum_{l=-\infty}^{\infty} x(l) \cdot Z^{-(n-K)}$

24-2-2020

Digital signal processing

Que:- $x(n) = \cos wn \cdot u(n)$, find Z-Transform.

Soln:- $x(n) = \cos wn \cdot u(n)$

$$= \frac{1}{2}\left[e^{jwn} + e^{-jwn}\right] u(n)$$

$$= \frac{1}{2} \cdot e^{jwn} \cdot u(n) + \frac{1}{2} \cdot e^{-jwn} \cdot u(n)$$

$$= \frac{1}{2} x_1(n) + \frac{1}{2} x_2(n)$$

$$X(Z) = \frac{1}{2} \cdot X_1(Z) + X_2(Z) \cdot \frac{1}{2}$$

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} e^{jwn} \cdot u(n) \cdot Z^{-n}$$

$u(n)=1$; for $n \geq 0$

$$= \sum_{n=0}^{\infty} e^{jwn} \cdot 1 \cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} \Big(\underbrace{e^{jw} \cdot Z^{-1}}_{a}\Big)^n$$

$1 + a + a^2 + a^3 + \cdots \infty$

$= \frac{1}{1-a}$

$$= \frac{1}{1 - e^{jw} \cdot Z^{-1}}$$

$$X_2(Z) = \sum_{n=0}^{\infty} e^{-jwn} \cdot Z^{-n} = \sum_{n=0}^{\infty} \left(e^{-jw} \cdot Z^{-1}\right)^n$$

$$= \boxed{\frac{1}{1 - e^{-jw} \cdot Z^{-1}}}$$

$$X(Z) = \frac{1}{2} \cdot X_1(Z) + \frac{1}{2} \cdot X_2(Z)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{1 - e^{j\omega} \cdot Z^{-1}} + \frac{1}{1 - e^{-j\omega} \cdot Z^{-1}} \right]$$

$$= \frac{1}{2} \cdot \left[\frac{1 - e^{-j\omega} \cdot Z^{-1} + 1 - e^{j\omega} \cdot Z^{-1}}{(1 - e^{j\omega} \cdot Z^{-1})(1 - e^{-j\omega} \cdot Z^{-1})} \right]$$

$$= \frac{1}{2} \cdot \left[\frac{2 - e^{j\omega} Z^{-1} - e^{-j\omega} Z^{-1}}{1 - e^{-j\omega} \cdot Z^{-1} - e^{j\omega} \cdot Z^{-1} + e^{j\omega} \cdot Z^{-1} \cdot e^{-j\omega} \cdot Z^{-1}} \right]$$

$$= \frac{1}{2} \cdot 2 \left[\frac{1 - \frac{Z^{-1}(e^{j\omega} + e^{-j\omega})}{2}}{1 - Z^{-1}(e^{j\omega} + e^{-j\omega}) + Z^{-2}} \right]$$

$$e^{j0} \cdot e^{j0} = 1$$

$$Z^{-1} \cdot Z^{-1} = Z^{-2}$$

$$= \left[\frac{1 - Z^{-1} \cdot \cos\omega}{1 - Z^{-1} \cdot 2 \cdot \frac{(e^{j\omega} + e^{-j\omega})}{2} + Z^{-2}} \right]$$

$$\cos\omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$$

$$= \left[\frac{1 - Z^{-1} \cdot \cos\omega}{1 - Z^{-1} \cdot 2 \cdot \cos\omega + Z^{-2}} \right]$$

$$= \left[\frac{1 - Z^{-1} \cdot \cos\omega}{1 - 2Z^{-1} \cdot \cos\omega + Z^{-2}} \right]$$

$$= Z\left[\cos\omega n \cdot u(n) \right]$$

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TIME SHIFTING

If $Z[x(n)] = X(Z)$ then

$$Z[x(n-K)] = Z^{-K} \cdot X(Z)$$

$$Z[x(n+K)] = Z^{K} \cdot X(Z)$$

Que:- $x_1(n) = \{ \underset{\uparrow}{1}, 2, 3, 4, 5 \}$

$$X(Z) = \sum_{n=0}^{4} x_1(n) \cdot Z^{-n}$$

$$= x_1(0) \cdot Z^{-0} + x_1(1) \cdot Z^{-1} + x_1(2) \cdot Z^{-2} + x_1(3) \cdot Z^{-3} + x_1(4) \cdot Z^{-4}$$

$$= 1 \cdot Z^{-0} + 2 \cdot Z^{-1} + 3 \cdot Z^{-2} + 4 \cdot Z^{-3} + 5 \cdot Z^{-4}$$

$$= 1 + 2Z^{-1} + 3Z^{-2} + 4 \cdot Z^{-3} + 5 \cdot Z^{-4}$$

ROC: Entire Z-plane except $Z = 0$

$x_2(n) = \{ 1, 2, \underset{\substack{\uparrow \\ n=0}}{3}, 4, 5 \}$

$= x_1(n+2)$

$$Z[x_1(n+2)] = Z^{2} \cdot X_1(Z)$$

$$X_2(Z) = Z[x_1(n+2)]$$

$$= Z^2 \cdot [1 + 2Z^{-1} + 3Z^{-2} + 4 \cdot Z^{-3} + 5 \cdot Z^{-4}]$$

$$= Z^2 + 2 \cdot Z^1 + 3 + 4 \cdot Z^{-1} + 5 \cdot Z^{-2}$$

ROC:- Entire Z-plane except $Z = 0$ and $Z = \infty$.

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$x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 3, 4, 5\}$

$= x_1(n-2)$

$X_3(z) = Z[x_3(n)] = Z[x_1(n-2)]$

$= z^{-2} \cdot X_1(z)$

$= z^{-2} \cdot [1 + 2z^{-1} + 3z^{-2} + 4 \cdot z^{-3} + 5 \cdot z^{-4}]$

$= z^{-2} + 2 \cdot z^{-3} + 3z^{-4} + 4 \cdot z^{-5} + 5 \cdot z^{-6}$

ROC :- Entire z-plane except $z=0$.

---x---x---x---

TIME REVERSAL :-

If $x(n) \xleftrightarrow{Z} X(z)$, ROC: $r_1 < |z| < r_2$

Then

$x(-n) \xleftrightarrow{Z} X(z^{-1})$, ROC :

$\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Que :- $x(n) = a^{-n} \cdot u(-n)$, find Z-Transform by using Time reversal property.

Soln :- $Z[\underbrace{a^n \cdot u(n)}_{x_1(n)}] = \frac{1}{1 - a \cdot z^{-1}}$

$= X_1(z)$ ROC is $|z| > a$

$Z[x_1(-n)] = X_1(z^{-1})$

$$Z\left[a^{-n} \cdot u(-n)\right] = \frac{1}{1 - a \cdot (z^{-1})^{-1}}$$

$$= \boxed{\frac{1}{1 - a \cdot z}}$$

$\downarrow$ ROC: $|z| < \frac{1}{a}$

x — x

SCALING IN Z-DOMAIN:-

$\rightarrow$ ROC: $r_1 < |z| < r_2$

If $Z[x(n)] = X(z)$ Then

$$Z\left[a^n \cdot x(n)\right] = X(a^{-1}z) \rightarrow \text{ROC: } |a| r_1 < |z| < |a| r_2$$

x

Que:- If $x(n) = a^n \cdot \sin w_0 n \cdot u(n)$ Then find its Z-Transform.

Sol^n:- $$Z\left[\sin w_0 n \cdot u(n)\right] = \frac{z^{-1} \cdot \sin w_0}{1 - 2z^{-1} \cdot \cos w_0 + z^{-2}}$$

$$Z\left[a^n \cdot \sin w_0 n \cdot u(n)\right]$$

$$= \frac{(a^{-1}z)^{-1} \cdot \sin w_0}{1 - 2(a^{-1}z)^{-1} \cdot \cos w_0 + (a^{-1}z)^{-2}}$$

$$= \frac{a \cdot z^{-1} \cdot \sin w_0}{1 - 2a \cdot z^{-1} \cdot \cos w_0 + a^2 \cdot z^{-2}}$$

$$\sin w_0 n = \frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}$$

$$Z[\sin\omega_0 n \cdot u(n)] = \sum_{n=-\infty}^{\infty} \sin\omega_0 n \cdot \underbrace{u(n)}_{1} \cdot z^{-n}$$

$u(n) = 1;\ n \geq 0$
$= 0;\ n < 0$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \right) \cdot z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{j\omega_0 n} \cdot z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} \cdot z^{-n} \right]$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} (\underbrace{e^{j\omega_0} \cdot z^{-1}}_{a})^{n} - \sum_{n=0}^{\infty} (\underbrace{e^{-j\omega_0} \cdot z^{-1}}_{b})^{n} \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} \cdot z^{-1}} - \frac{1}{1 - e^{-j\omega_0} \cdot z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{1 - e^{-j\omega_0} \cdot z^{-1} - (1 - e^{j\omega_0} \cdot z^{-1})}{(1 - e^{j\omega} \cdot z^{-1})(1 - e^{-j\omega} \cdot z^{-1})} \right]$$

$$= \frac{1}{2j} \left[\frac{1 - e^{-j\omega} \cdot z^{-1} - 1 + e^{j\omega} \cdot z^{-1}}{1 - e^{-j\omega} \cdot z^{-1} - e^{j\omega} \cdot z^{-1} + e^{j\omega} \cdot z^{-1} \cdot e^{-j\omega} \cdot z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{z^{-1} e^{j\omega} - z^{-1} e^{-j\omega}}{1 - z^{-1}(e^{j\omega} + e^{-j\omega}) + z^{-2}} \right]$$

$$= \frac{z^{-1} \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)}{1 - z^{-1} \cdot 2 \cdot \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + z^{-2}}$$

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$$= \frac{z^{-1} \cdot \sin \omega_0}{1 - z^{-1} \cdot 2 \cdot \cos \omega_0 + z^{-2}}$$

$$= \boxed{\frac{z^{-1} \cdot \sin \omega_0}{1 - 2z^{-1} \cdot \cos \omega_0 + z^{-2}}}$$

—— x ——

Differentiation in Z-Domain:-

If $Z[x(n)] = X(Z)$ Then

$$Z[n \cdot x(n)] = -Z \cdot \frac{dX(Z)}{dZ}$$

Que:- using Differentiation property Determine the Z-Transform of the signals $x(n) = n \cdot u(n)$

Sol^n:- $Z[a^n \cdot u(n)] = \frac{1}{1 - aZ^{-1}}$

$$Z[(1)^n \cdot u(n)] = \frac{1}{1 - 1 \cdot Z^{-1}}$$

$$\Rightarrow Z[u(n)] = \frac{1}{1 - Z^{-1}} = \frac{1}{1 - \frac{1}{Z}}$$

$$= \frac{1}{\frac{Z-1}{Z}} = \boxed{\frac{Z}{Z-1}}$$

$$Z[n \cdot u(n)] = -Z \cdot \frac{d}{dZ}\left(\frac{Z}{Z-1}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$u' = \frac{du}{dx}, \quad v' = \frac{dv}{dx}$$

$$= -Z\left[\frac{\frac{dZ}{dZ}\cdot(Z-1) - Z\cdot\frac{d(Z-1)}{dZ}}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{(Z-1) - Z\cdot 1}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{\cancel{Z}-1-\cancel{Z}}{(Z-1)^2}\right]$$

$$= \frac{Z}{(Z-1)^2}$$

xxx xxx

Que:- Deteremine the signal $x(n)$ where Z-Transform is given by

$$X(Z) = \log(1+aZ^{-1}), \quad |Z|>|a|$$

Sol^n:-
$$\frac{dX(Z)}{dZ} = \frac{d\log(1+aZ^{-1})}{d(1+aZ^{-1})}\cdot\frac{d(1+aZ^{-1})}{dZ}$$

$$= \frac{1}{(1+aZ^{-1})}\cdot a\cdot\frac{dZ^{-1}}{dZ}$$

$$= a\cdot\frac{1}{(1+aZ^{-1})}\cdot(-1)\cdot Z^{-1-1}$$

$$= -a \cdot \frac{1}{(1+az^{-1})} \cdot z^{-2}$$

$$-z \cdot \frac{dX(z)}{dz} = \frac{-z \cdot (-a) \cdot z^{-2}}{(1+az^{-1})}$$

$$= \frac{z^{-1} \cdot a}{1+az^{-1}}$$

$$= a \cdot z^{-1} \cdot \frac{1}{1-(-a)z^{-1}}$$

Take Inverse Z-Transform on both sides

$$a^n \cdot u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$$

$$(-a)^n u(n) \xleftrightarrow{Z} \frac{1}{1-(-a)z^{-1}}$$

$$(-a)^{n-1} u(n-1) \xleftrightarrow{Z} \frac{z^{-1}}{1+az^{-1}}$$

$$n \cdot x(n) = a \cdot (-a)^{n-1} \cdot u(n-1)$$

$$\Rightarrow x(n) = \frac{1}{n} \cdot a^1 \cdot (-1)^{n-1} \cdot (a)^{n-1} \cdot u(n-1)$$

$$= \frac{1}{n} \cdot a^n \cdot (-1)^{n-1} \cdot u(n-1)$$

Convolution property :-

If Two signals $x_1(n)$ and $x_2(n)$ and we will make convolution

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X(Z) = X_1(Z) \cdot X_2(Z)$$

$$Z[x_1(n) * x_2(n)] = X_1(Z) \cdot X_2(Z)$$

That is convolution of Two signals in Time Domain is product of Two signals in Z-Domain

~~The~~ The Region of convergence (ROC) of $X(Z)$ is the intersection of $X_1(Z)$ and $X_2(Z)$.

Que:- Find the convolution of signal $x_1(n) = a^n.u(n)$, $x_2(n) = u(n)$ using Z-Transform.

Sol^n:- $X_1(Z) = Z[x_1(n)] = Z[a^n.u(n)]$

$$= \frac{1}{1-a.Z^{-1}},$$

ROC: $|Z| > |a|$

$X_2(Z) = Z[u(n)] = \frac{1}{1-Z^{-1}}$, ROC: $|Z| > |1|$

$x(n) = x_1(n) * x_2(n)$

$Z[x(n)] = X(Z) = Z[x_1(n) * x_2(n)]$

$$= X_1(Z) \cdot X_2(Z)$$

$\Rightarrow X(Z) = \frac{1}{1-aZ^{-1}} \cdot \frac{1}{1-Z^{-1}}$, By making partial fraction

$$= \frac{A}{1-aZ^{-1}} + \frac{B}{1-Z^{-1}}$$

$$A = \lim_{Z^{-1} \to \frac{1}{a}} (1-aZ^{-1}) \cdot X(Z)$$

$$= \lim_{Z^{-1} \to \frac{1}{a}} \cancel{(1-aZ^{-1})} \cdot \frac{1}{\cancel{(1-aZ^{-1})}(1-Z^{-1})}$$

$$= \frac{1}{1-\frac{1}{a}} = \frac{1}{\frac{a-1}{a}} = \frac{a}{a-1}$$

$$B = \lim_{Z^{-1}\to 1} (1-Z^{-1})\cdot X(Z)$$

$$= \lim_{Z^{-1}\to 1} \cancel{(1-Z^{-1})}\cdot \frac{1}{(1-aZ^{-1})\cancel{(1-Z^{-1})}}$$

$$= \frac{1}{1-a\times 1} = \frac{1}{1-a}$$

$$X(Z) = \frac{a/a-1}{1-aZ^{-1}} + \frac{1/1-a}{1-aZ^{-1}}$$

$$= \frac{-\frac{a}{1-a}}{1-aZ^{-1}} + \frac{1/1-a}{1-Z^{-1}}$$

$$= \frac{1}{1-a}\left[\frac{1}{1-Z^{-1}} - \frac{a}{1-aZ^{-1}}\right]$$

$=$

Making Inverse Z-Transform

$$Z^{-1}[X(Z)] = x(n)$$

$$= \frac{1}{1-a}\cdot\left[u(n) - a\cdot a^{n}\cdot u(n)\right]$$

$$\Rightarrow x(n) = \frac{1}{1-a}\left[u(n) - a^{n+1}\cdot u(n)\right]$$

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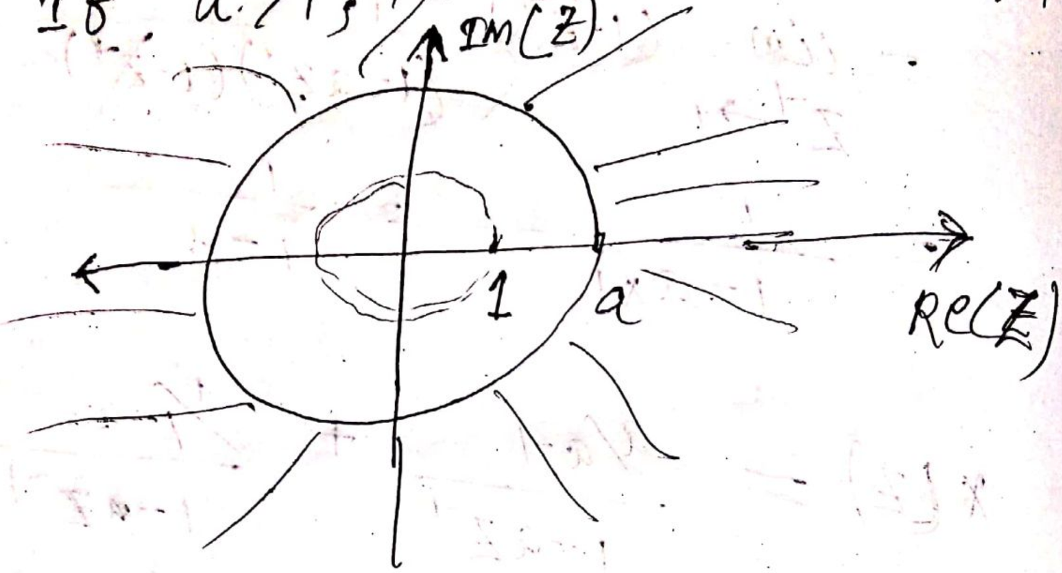
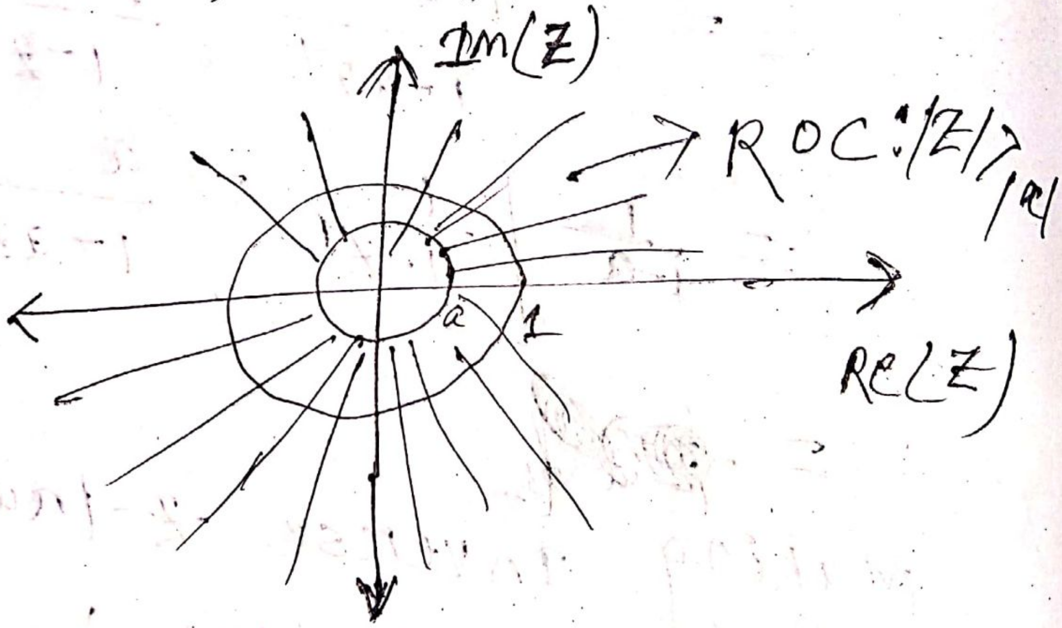
ROC : $R_1 \cap R_2$

$|Z| > |a|$ $|z| > 1$

If $a > 1$, Then ROC R: $|Z| > |a|$



If $a < 1$, Then ROC R: $|Z| > |a|$



The convolution property is most powerful property of the Z-Transform.

— x —

Initial Value Theorem:-

If $x(n) = 0$ for $n < 0$, Then

$$x(0) = \lim_{z \to \infty} X(z)$$

proof:-

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$\Rightarrow X(z) = x(0) \cdot z^{-0} + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + \cdots$$

$$= x(0) \cdot 1 + x(1) \cdot \frac{1}{z} + x(2) \frac{1}{z^2} + \cdots$$

$$\lim_{z \to \infty} X(z) = x(0) + \frac{x(1)}{\infty} + x(2) \frac{1}{\infty} + \cdots$$

$$= x(0) + 0 + 0 + 0 \cdots$$

$$= x(0)$$

Correlation of Two sequences:-

If Two signals $x_1(n)$ and $x_2(n)$ Then Their correlation is

$$r_{x_1 x_2}(n) = x_1(n) * x_2(-n)$$

$$\updownarrow Z \qquad \qquad \updownarrow Z \longrightarrow \text{Z-Transform}$$

$$R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

The ROC of $R_{x_1 x_2}$ is the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$.

Que:- Find the Z-Transform and ROC of the signal sequence.

$$x(n) = [4(2^n) - 5(3^n)]u(n)$$

Sol^n:- $x(n) = 4.(2)^n.u(n) - 5(3)^n.u(n)$

$= 4.x_1(n) - 5.x_2(n)$

$x_1(n) = (2)^n.u(n)$

$X_1(Z) = Z[(2)^n.u(n)] = \frac{1}{1-2Z^{-1}}$

~~$X_1(Z) = Z[$~~ ROC $R_1 : |Z| > |2|$

$X_2(Z) = Z[(3)^n.u(n)] = \frac{1}{1-3Z^{-1}}$

ROC $R_2 : |Z| > |3|$

ROC $R : R_1 \cap R_2$

$|Z| > |2|, \quad |Z| > |3|$

$$\boxed{\Rightarrow \text{ROC } R : |Z| > |3| \text{ for } X(Z)}$$

$X(Z) = Z[x(n)]$

$= 4 \cdot \frac{1}{1-2 \cdot Z^{-1}} - 5 \cdot \frac{1}{1-3Z^{-1}}$

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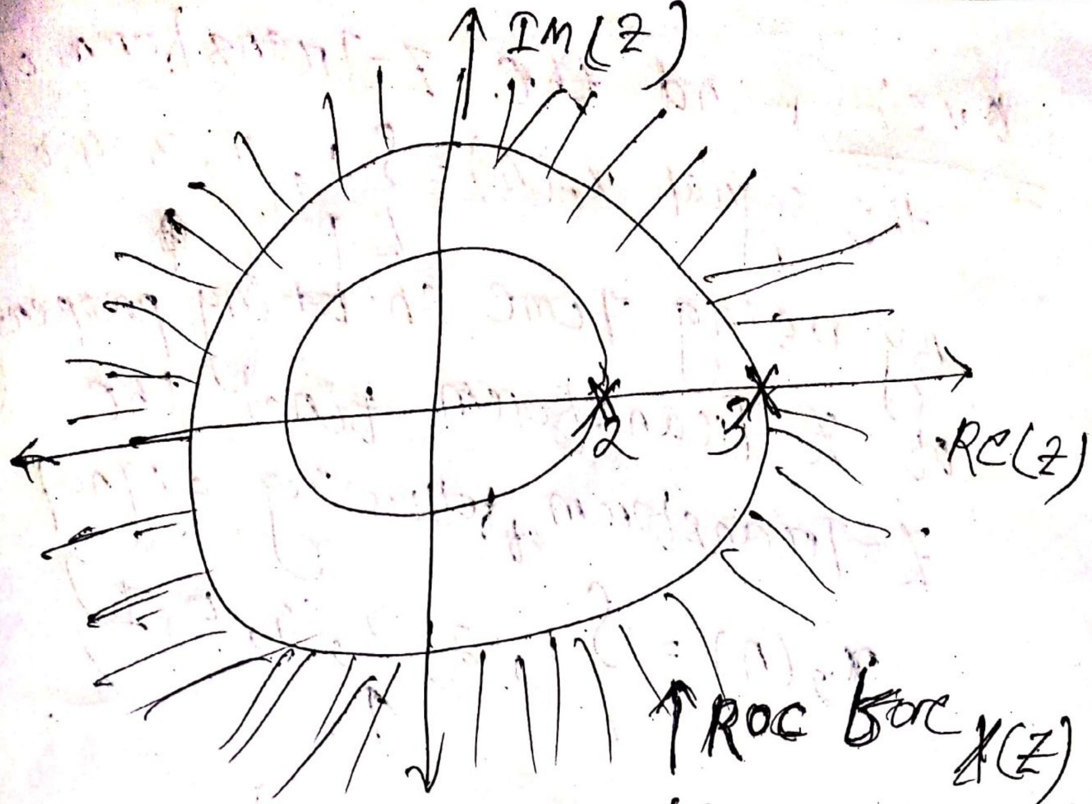

ROC for $X(Z)$

is $R : R_1 \cap R_2$

$: |Z| > |3|$

Que:- Find the Z-Transform of sequence $x(n) = u(-n)$

Sol:- we know $Z[x(n)] = X(Z)$

and $Z[x(-n)] = X(Z^{-1})$

Similary we know the Z-Transform of $u(n)$ is

$$Z[u(n)] = \frac{1}{1-z^{-1}}$$

and $$Z[u(-n)] = \frac{1}{1-(Z^{-1})^{-1}}$$

$$= \frac{1}{1+Z} \; ; \; ROC : |Z| > 1$$

Que:- Find the Z-Transform of the signal $x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 0, 1\}$.

By using Time shifting property of Z-Transform find the Z-Transform of following signal

$$x_2(n) = \{1, 2, \underset{\uparrow}{3}, 4, 0, 1\}$$

Sol^n:- $x_2(n) = x_1(n+2)$

$$X_2(z) = Z[x_1(n+2)] = z^2 \cdot X_1(z) \quad \text{---(1)}$$

$$X_1(z) = \sum_{n=0}^{5} x(n) \cdot z^{-n}$$

$$= x(0) \cdot z^{-0} + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + x(3) \cdot z^{-3} + x(4) \cdot z^{-4} + x(5) \cdot z^{-5}$$

$$= 1 \cdot z^{-0} + 2 \cdot z^{-1} + 3 \cdot z^{-2} + 4 \cdot z^{-3} + 0 \cdot z^{-4} + 1 \cdot z^{-5}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + z^{-5}$$

ROC: $R_1 \rightarrow$ Entire z-plane except $z = 0$

From Eq^n-(1)

$$X_2(z) = z^2 \cdot X_1(z)$$

$$= z^2 \cdot [1 + 2z^{-1} + 3z^{-2} + 4 \cdot z^{-3} + z^{-5}]$$

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$$= Z^{2} + 2\cdot Z^{1} + 3\cdot Z^{0} + 4\cdot Z^{-1} + Z^{-3}$$

ROC : R2 → Entire z-plane except $z=0$ and $z=\infty$.

Que :- Find the convolution of following two sequencies by using Z-transform Property.

$x_1(n) = \{\underset{\uparrow}{1}, -2, 1\}$, $x_2(n) = \{\underset{\uparrow}{1}, 1, 1\}$

A) $x_1(z) = Z[x_1(n)] = \sum_{n=0}^{2} x_1(n).z^{-n}$

$= x_1(0).z^{-0} + x_1(1).z^{-1} + x_1(2).z^{-2}$

$= 1.z^{-0} + (-2).z^{-1} + 1.z^{-2}$

$= 1 - 2.z^{-1} + z^{-2}$

$x_2(z) = Z[x_2(n)] = \sum_{n=0}^{2} x_2(n) z^{-n}$

$= x_2(0).z^{-0} + x_2(1).z^{-1} + x_2(2).z^{-2}$

$= 1.z^{-0} + 1.z^{-1} + 1.z^{-2} = 1 + z^{-1} + z^{-2}$

Take $x(n) = x_1(n) * x_2(n)$

By making z-transform on both sides.

$\Rightarrow Z[x(n)] = Z[x_1(n) * x_2(n)] = x_1(z).x_2(z)$

$\Rightarrow x(z) = (1 - 2z^{-1} + z^{-2}).(1 + z^{-1} + z^{-2})$

$\Rightarrow x(z) = 1 + z^{-1} + z^{-2} - 2z^{-1} - 2z^{-2} - 2z^{-3} + z^{-2} + z^{-3} + z^{-4}$

$\Rightarrow x(z) = 1 - z^{-1} + 0.z^{-2} - z^{-3} + z^{-4} \rightarrow (1)$

By taking inverse z transform to above equation.

$\Rightarrow x(n) = \{1, -1, 0, -1, 1\}$.

Que Find the z-transform of the sequence

$x(n) = \left(\frac{1}{3}\right)^{n-1}.u(n-1)$.

A) We know $Z[x(n-k)] = z^{-k}.x(z)$

$\because Z(x(n+k)] = Z^k \cdot x(z)$

$Z\left[\left(\frac{1}{3}\right)^n \cdot u(n)\right] = \frac{1}{1-\frac{1}{3}\cdot Z^{-1}}$

$\therefore Z[x(n)] = X(z) = Z\left[\left(\frac{1}{3}\right)^{n-1} \cdot u(n-1)\right]$

$= Z^{-1} \cdot \frac{1}{1-\frac{1}{3}\cdot Z^{-1}}$

Que :- Find the Z-transform of the sequence.

$x(n) = n \cdot a^n \cdot u(n)$

A) We know if $x(n) \overset{Z}{\longleftrightarrow} X(z)$

Then $n \cdot x(n) \overset{Z}{\longleftrightarrow} -Z \cdot \frac{dX(z)}{dz}$

Here,

$x(n) = n \cdot s(n)$, where $s(n) = a^n \cdot u(n)$

$S(z) = Z[s(n)] = Z[a^n \cdot u(n)]$

$= \frac{1}{1-a\cdot Z^{-1}} = \frac{1}{1-a\cdot\frac{1}{Z}} = \frac{1}{\frac{Z-a}{Z}} = \frac{Z}{Z-a}$

Now making Z transform of $\cancel{x(n)}$

$Z[x(n)] = X(z) = Z[n \cdot s(n)]$

$= -Z \cdot \frac{dS(z)}{dZ} \rightarrow (1)$

$\because \frac{d}{dx}\left(\frac{U}{V}\right) = \frac{U'V - U \cdot V'}{V^2}$

$U' = \frac{dU}{dx}, V' = \frac{dV}{dx}$

$\Rightarrow X(z) = -Z \cdot \frac{d}{dZ}\left(\frac{Z}{Z-a}\right)$

$\Rightarrow X(z) = -Z\left[\frac{\frac{dz}{dZ}(Z-a) - Z \cdot \frac{d(Z-a)}{dZ}}{(Z-a)^2}\right]$

$\Rightarrow X(z) = -Z\left[\frac{1(Z-a) - Z \cdot 1}{(Z-a)^2}\right]$

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$$= -z\left[\frac{z-a-z}{(z-a)^2}\right] \Rightarrow X(z) = \frac{a \cdot z}{(z-a)^2}$$

Que :- Find the System function & impulse response of the system described by the difference eqn.

$y(n) = \frac{1}{5} y(n-1) + x(n)$

A) given, system is $y(n) = \frac{1}{5} y(n-1) + x(n)$

Take z transform on both sides.

$\Rightarrow Z[y(n)] = \frac{1}{5} Z[y(n-1)] + Z[x(n)]$

$\Rightarrow Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$

$\Rightarrow Y(z) - \frac{1}{5} z^{-1} Y(z) = X(z)$

$\Rightarrow Y(z) \cdot \left[1 - \frac{1}{5} z^{-1}\right] = X(z)$

$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5} z^{-1}}$

$\Rightarrow H(z) = \text{system function} = \frac{1}{1 - \frac{1}{5} \cdot z^{-1}}$

we will find inverse z-transform is

Impulse response $= h(n) = \left(\frac{1}{5}\right)^n \cdot u(n)$

Ques :- $y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

A) Here $x(n)$ is i/p & $y(n)$ is o/p

The given system is

$y(n) = x(n) + 2x(n-1) - 4x(n-2) + x(n-3)$

Take z-transform on both sides we will get,

$\Rightarrow Y(z) = X(z) + 2z^{-1} X(z) - 4 \cdot z^{-2} X(z) + z^{-3} X(z)$

$\Rightarrow Y(z) = X(z)\left[1 + 2 \cdot z^{-1} - 4 \cdot z^{-2} + z^{-3}\right]$

$\Rightarrow \frac{Y(z)}{X(z)} = \cancel{X(z)} \left(1 + 2 \cdot z^{-1} - 4 \cdot z^{-2} + z^{-3}\right] = H(z) = \text{System function}$

By making inverse z transform on both sides we will get

$\Rightarrow h(n) = \{\underset{\uparrow}{1}, 2, -4, 1\}$ = Impulse response of the system.

Que Find the pole-zero plot for the system described by the difference eqn

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

$\Rightarrow$ Given, difference eqn is

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Taking z-transform on both sides, we get

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

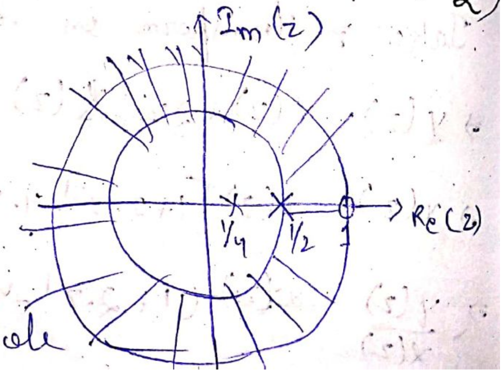
$$\Rightarrow Y(z)\cdot\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\cdot\left[1 - z^{-1}\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1 - z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) + \frac{1}{8}z^{-2} - \frac{1}{4}z^{-1}}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) + \frac{1}{4}z^{-1}\left(\frac{1}{2}z^{-1} - 1\right)} = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) - \frac{1}{4}z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$\Rightarrow H(z) = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$



ROC: $|z| > \frac{1}{2}$

Outerside of outermost pole

of system function
Here, ROC includes the unit circle; Hence the system is stable.

⇒ Here ROC of system function cannot contain any pole.

Que :- Find the Z-transform & ROC of given signal
$x(n) = \frac{1}{2}\cdot\delta(n) + \delta(n-1) - \frac{1}{3}\delta(n-2)$

⇒ $X(z) = Z[x(n)]$

$= \frac{1}{2} Z[\delta(n)] + Z[\delta(n-1)] - \frac{1}{3}[\delta(n-2)]$

$= \frac{1}{2}\cdot 1 + z^{-1}\cdot Z[\delta(n)] - \frac{1}{3}\cdot z^{-2}\cdot Z[\delta(n)]$

$= \frac{1}{2} + z^{-1} - \frac{1}{3}\cdot z^{-2}$

ROC. Entire Z-Plane except $z = 0$.

Note :- $Z[\delta(n)] = 1$: unit impulse signal.

Que :- $x(n) = u(n-2)$

⇒ $X(z) = Z[x(n)] = Z[u(n-2)]$

$= z^{-2}\cdot Z[u(n)] = z^{-2}\cdot\frac{1}{1-z^{-1}} = z^{-2}\cdot\frac{1}{1-\frac{1}{z}}$

$= z^{-2}\cdot\frac{1}{\frac{z-1}{z}} = z^{-2}\frac{z^{-1}}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$

Here ROC : $|z| > 1$

Que :- $x(n) = (n+0.5)\frac{1}{3}^{n}\cdot u(n)$

⇒ $X(z) = Z[x(n)]$ $= \cancel{Z\left[(n+0.5)\frac{1}{3}^{n}\cdot u(n)\right]}$

$= Z\left[n\cdot\left(\frac{1}{3}\right)^{n}\cdot u(n) + 0.5\left(\frac{1}{3}\right)^{n} u(n)\right]$

$$= Z\left[n\cdot\left(\frac{1}{3}\right)^n u(n)\right] + 0.5\, Z\left[\left(\frac{1}{3}\right)^n\cdot u(n)\right]$$

$$= Z\left[n\cdot x_1(n)\right] + 0.5 \times \frac{1}{1-\frac{1}{3}z^{-1}}$$

NOTE :- $Z\left[a^n\cdot u(n)\right] = \frac{1}{1-a\cdot z^{-1}} = \frac{z}{z-a}$

$$= -z\,\frac{dx_1(z)}{dz} + 0.5\,\frac{z}{z-\frac{1}{3}} \rightarrow (1).$$

$$\Rightarrow \frac{dx_1(z)}{dz} = \frac{d}{dz}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$\Rightarrow \frac{dx_1(z)}{dz} = \frac{\frac{dz}{dz}\cdot\left(z-\frac{1}{3}\right) - z\cdot\frac{d\left(z-\frac{1}{3}\right)}{dx}}{\left(z-\frac{1}{3}\right)^2}$$

$$= \frac{\left(z-\frac{1}{3}\right) - z\cdot 1}{\left(z-\frac{1}{3}\right)^2} = \frac{-1/3}{\left(z-\frac{1}{3}\right)^2}$$

$$\therefore x(z) = (-z)\frac{\left(-\frac{1}{3}\right)}{\left(z-\frac{1}{3}\right)^2} + \frac{0.5z}{z-\frac{1}{3}}$$

$$= \frac{z}{3\left(z-\frac{1}{3}\right)^2} + \frac{z}{2\left(z-\frac{1}{3}\right)}$$

Que (a) If $Y(z) = \frac{0.5\left(1-0.5z^{-1}\right)}{\left(1-0.25z^{-1}\right)\left(1-0.75z^{-1}\right)\left(1-z^{-1}\right)}$

Find the steady state value of $y(n)$ if it exists.

(b) Find $x(\infty)$ if $x(z)$ is given by $x(z) = \frac{3z}{(z-1)(z+1)}$

A) **Final Value Theorem** :-

According to final value Theorem,

$(z-1)x(z)$ or $(1-z^{-1})x(z)$ has all the Poles should lie inside the unit circle, then only final value of $x(n)$ will exist. That means no Pole should lie on the unit circle (or) outside the unit circle.

a) Steady state value of $y(n)$ is

$$y(\infty) = \lim_{z^{-1}\to 1} (1-z^{-1})\, Y(z)$$

$$\lim_{z^{-1}\to 1} (1-z^{-1}) \cdot \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})(1-z^{-1})}$$

Here two Poles 0.25 & 0.75 lie inside the unit circle.

$$= \lim_{z^{-1}\to 1} \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})}$$

$$= \frac{0.5\times(1-0.5\times1)}{(1-0.25\times1)(1-0.75\times1)} = \frac{0.5\times0.5}{0.75\times0.25}$$

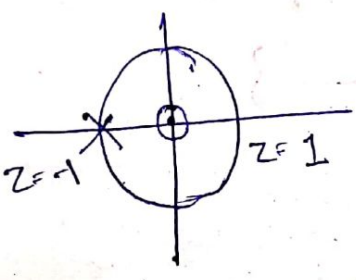
$$= \frac{0.25}{0.75\times0.25} = \frac{1}{0.75} = 1.33$$

d) Steady state value of $x(n)$ is

$$x(\infty) = \lim_{z\to 1} (z-1)\cdot X(z)$$

$$= \lim_{z\to 1} (z-1)\frac{3z}{(z-1)(z+1)} = \lim_{z\to 1} \frac{3z}{z+1}$$

Here $(z-1)X(z) = \frac{3z}{z+1}$ has one Pole at $z=-1$ on the unit circle. So final value of $x(\infty)$ does not exist.



* The Inverse Z-Transform :-

→ It is expressed as $x(n) = Z^{-1}[X(z)]$.

By using 3 methods we can perform the inverse Z-transform.

① Long Division Method

② Partial Fraction Expansion Method

③ Residue method

① Long Division Method :-

Que Determine the inverse Z-transform of

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$ when

(i) ROC : $|z| > 1$; (ii) ROC : $|z| < \frac{1}{2}$.

A)

(i) Here ROC is $|z| > 1$; that means outwards of the unit circle.

So, $x(n)$ is causal signal

$$\therefore X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \ldots$$

By taking Inverse Z-transform, we will get

$$x(n) = \{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \ldots\}$$

$n = 0$

$$
\begin{array}{r|l}
 & 1 + \tfrac{3}{2}z^{-1} + \tfrac{7}{4}z^{-2} + \cdots \\
1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} & 1 \\
 & 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\
 & (-)\ (+)\ (-) \\
 & \frac{3}{2}z^{-1} - \frac{1}{2}z^{-2} \\
 & \frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3} \\
 & (-)\ (+)\ (-) \\
 & \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} + \frac{7}{8}z^{-4} \\
 & (-)\ (+)\ (-) \\
 & \frac{15}{8}z^{-3} - \frac{7}{8}z^{-4}
\end{array}
$$

(ii) In this case ROC is $|z| < 0.5$; that means the interior of the circle. Here $x(n)$ signal is anti-causal signal. Here we will get a power series expansion is +ve powers of z. We perform the long division method in following way:-

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + \cdots \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & 1 - 3z + 2z^2 \\
 & (-)\ (+)\ (-) \\
 & 3z - 2z^2 \\
 & 3z - 9z^2 + 6z^3 \\
 & (-)\ (+)\ (-) \\
 & 7z^2 - 6z^3 \\
 & 7z^2 - 21z^3 + 14z^4 \\
 & (-)\ (+)\ (-) \\
 & 15z^3 - 14z^4
\end{array}
$$

$$\therefore X(z) = \frac{1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1}$$

$$= 0 + 0.z^{-1} + 2z^2 + 6z^3 + 14z^4 + \cdots$$

Take inverse Z-transform is

$x(n) = \{\ldots\ldots 14, 6, 2, 0, 0\}$
$\uparrow$ $n=0$

Anti-causal signal (or) left side signal

<u>28/2</u>

<u>Que</u> :- The impulse response for a discrete time system is given as $h(n) = \{1, 2, 3\}$ & o/p response is given as $y(n) = \{1, 1, 2, -1, 3\}$. Determine discrete time i/p signal.

A)
<u>Method 1</u>

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$y(n) = h(n) * x(n)$

Let $x(n) = \{a, b, c\}$.

$x(n)$ \ $h(n)$	1	2	3
a	a	2a	3a
b	b	2b	3b
c	c	2c	3c

$y(n) = \{a, b+2a, 3a+2b+c, 2c+3b, 3c\}$
$= \{1, 1, 2, -1, 3\}$

By comparing, $a = 1$; $b+2a$
$\Rightarrow b + 2.1 = 1 \Rightarrow b = -1.$; $c = 1$

$\therefore x(n) = \{a, b, c\} = \{1, -1, 1\}$.

<u>Method 2</u>

$y(n) = x(n) * h(n)$.

Make Z transform on both sides.

$\Rightarrow Y(z) = X(z) \cdot H(z)$.

$\Rightarrow X(z) = \dfrac{Y(z)}{H(z)}$ $\quad\therefore H(z) = \sum_{n=0}^{2} h(n) \cdot z^{-n}$.

$= 1 + 2 \cdot z^{-1} + 3 \cdot z^{-2}$.

$\therefore Y(z) = \sum_{n=0}^{4} y(n) \cdot z^{-n} = 1 + z^{-1} + 2 \cdot z^{-2} - z^{-3} + 3 \cdot z^{-4}$.

$$
\begin{array}{r|l}
 & 1 - z^{-1} + z^{-2} \\
\hline
1 + 2z^{-1} + 3z^{-2} & 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} \\
 & 1 + 2z^{-1} + 3z^{-2} \\
 & (-)\ \ (-)\ \ (-) \\
\hline
 & -z^{-1} - z^{-2} - z^{-3} + 3z^{-4} \\
 & -z^{-1} - 2z^{-2} - 3z^{-3} \\
 & (+)\ \ (+)\ \ (+) \\
\hline
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & (-)\ \ (-)\ \ (-) \\
\hline
 & 0
\end{array}
$$

$\therefore X(z) = 1 - z^{-1} + z^{-2}$

By making inverse z-transform

$x(n) = \{1, -1, 1\}$

② Inverse z-transform by using Partial Fraction Method :-

→ ~~If H(z) can~~ Here factorization is done in denominator.

→ If $H(z)$ can be written as $H(z) = \frac{A_1}{z-p} + \frac{A_2}{(z-p)^2} + \dots \frac{A_{m-1}}{(z-p)^{m-1}} + \frac{A_m}{(z-p)^m}$

Then $A_m = \lim_{z \to p} (z-p)^m . H(z)$

$$A_{m-1} = \frac{1}{1!} \lim_{z \to p} \frac{d^1}{dz^1} \left[(z-p)^m . H(z) \right]$$

$$A_{m-2} = \frac{1}{2!} \lim_{z \to p} \frac{d^2}{dz^2} \left[(z-p)^m . H(z) \right]$$

$$A_{m-3} = \frac{1}{3!} \lim_{z \to p} \frac{d^3}{dz^3} \left[(z-p)^m . H(z) \right]$$

$$A_3 = \frac{1}{(m-3)!} \lim_{z \to p} \cdot \frac{d^{m-3}}{dz^{m-3}} \left[(z-p)^{m} \cdot H(z) \right]$$

$$A_2 = \frac{1}{(m-2)!} \lim_{z \to p} \frac{d^{m-2}}{dz^{m-2}} \left[(z-p)^m \cdot H(z) \right]$$

$$A_1 = \frac{1}{(m-1)!} \lim_{z \to p} \frac{d^{m-1}}{dz^{m-1}} \left[(z-p)^m \cdot H(z) \right]$$

29/2

Que:- By using partial fraction method. find Inverse Z. transform of the following transfer function.

$$H(z) = \frac{-4+8 \cdot z^{-1}}{1+6z^{-1}+8z^{-2}}$$

A) Given transfer function is $H(z) = \frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$

$$= \frac{-4+8z^{-1}}{1+4z^{-1}+2z^{-1}+8z^{-2}} \quad = \frac{-4+8z^{-1}}{(1+4z^{-1}) + 2z^{-1}(1+4z^{-1})}$$

$$= \frac{-4+8z^{-1}}{(1+2z^{-1})(1+4z^{-1})}$$ (∵ By making Partial fraction)

$$\Rightarrow H(z) = \frac{A}{1+2z^{-1}} + \frac{B}{1+4z^{-1}}$$

$$A = \lim_{z^{-1} \to -1/2} (1+2z^{-1}) \cdot H(z)$$

$$= \lim_{z^{-1} \to -1/2} \cancel{(1+2z^{-1})} \cdot \frac{(-4+8z^{-1})}{\cancel{(1+2z^{-1})}(1+4z^{-1})}$$

$$= \frac{-4+8 \times (-1/2)}{1+4 \times (-1/2)} = \frac{-8}{-1} = 8.$$

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$$B = \lim_{z^{-1} \to -1/4} (1+4z^{-1}) . H(z)$$

$$= \lim_{z^{-1} \to -1/4} \cancel{1+4z^{-1}} . \frac{(-4+8z^{-1})}{(1+2z^{-1})\cancel{(1+4z^{-1})}}$$

$$= \frac{-4+8 \times (-1/4)}{1+2 \times (-1/4)} = \frac{-6}{1/2} = -12.$$

$$\therefore H(z) = \frac{8}{1+2z^{-1}} + \frac{-12}{1+4z^{-1}}$$

By making Inverse Z-transform,

$$h(n) = 8 . (-2)^n . u(n) - 12 (-4)^n . u(n)$$

Que By using Partial fraction method find inverse Z-transform. $x(z) = \dfrac{z^3}{(z+1)(z-1)^2}$

1) Given that, $\dfrac{x(z)}{z} = \dfrac{z^2}{(z+1)(z-1)^2}$.

$$\boxed{\begin{aligned} & Z[a^n . u(n)] = \frac{1}{1-az^{-1}} = \frac{Z}{Z-a} \\ & Z[(-a)^n . u(n)] = \\ & \frac{1}{1-(-a)z^{-1}} = \frac{1}{1+az^{-1}} \\ & = \frac{z}{z+a} \end{aligned}}$$

Take $F(z) = \dfrac{x(z)}{z} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

$$A = \lim_{z \to -1} (z+1) . F(z)$$

$$= \lim_{z \to -1} \cancel{(z+1)} . \frac{z^2}{\cancel{(z+1)}(z-1)^2} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$C = \lim_{z \to 1} (z-1)^2 . F(z)$$

$$= \lim_{z \to 1} \cancel{(z-1)^2} . \frac{z^2}{(z+1)\cancel{(z-1)^2}} = \frac{1^2}{1+1} = \frac{1}{2}$$

$$B = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 . F(z) \right]$$

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$$= \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \cdot \frac{z^2}{(z+1)(z-1)^2} \right]$$

$$\Rightarrow B = \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^2}{z+1}\right] \qquad \left[\because \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}\right]$$

$$= \lim_{z \to 1} \frac{\frac{dz^2}{dz}(z+1) - z^2 \cdot \frac{d(z+1)}{dz}}{(z+1)^2}$$

$$= \lim_{z \to 1} \frac{2 \cdot z(z+1) - z^2 \cdot 1}{(z+1)^2} = \lim_{z \to 1} \frac{2z^2 + 2z - z^2}{(z+1)^2}$$

$$= \lim_{z \to 1} \frac{z^2 + 2z}{(z+1)^2} = \frac{1^2 + 2 \times 1}{(1+1)^2} = 3/4$$

$$\therefore F(z) = \frac{x(z)}{z} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\Rightarrow \frac{x(z)}{z} = \frac{1}{4} \cdot \frac{1}{z+1} + \frac{3}{4} \cdot \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{(z-1)^2}$$

$$\Rightarrow x(z) = \frac{1}{4} \cdot \frac{z}{z+1} + \frac{3}{4} \cdot \frac{z}{z-1} + \frac{1}{2} \cdot \frac{z}{(z-1)^2}$$

Make inverse z-transform on both sides, we get,

$$\rightarrow x(n) = \frac{1}{4} \cdot (-1)^n u(n) + \frac{3}{4} \cdot (1)^n \cdot u(n) + \frac{1}{2} \cdot n(1)^n \cdot u(n)$$

$$\Rightarrow x(n) = \frac{1}{4}(-1)^n \cdot u(n) + \frac{3}{4} \cdot u(n) + \frac{1}{2} n \cdot u(n)$$

3.7.3 Residue Method

In this method, we obtain, inverse z-transform x [n], by summing residues of [X (z) z^{n-1}] at all poles. Mathematically, this may be expressed as

$$x(n) = \sum_{\substack{\text{all poles} \\ X(z)}} \text{residues of } (X(z)\, z^{n-1}) \qquad \ldots(3.46)$$

Here, the residue for any pole of order m at $z = \beta$ is

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to \beta} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-\beta)^m X(z).z^{n-1} \right] \right\} \qquad \ldots(3.47)$$

Example 3.35

Use residue method to find the inverse z-transform, x (n) for

$$X(z) = \frac{z}{(z-1)(z-2)}$$

Solution: The given transform is

$$X(z) = \frac{z}{(z-1)(z-2)}$$

X(z) has two poles of order m = I at z = 1 and at z = 2.

We can obtain the corresponding residues as ahead :

For poles at z = 1

$$\text{Residue} = \frac{1}{0!}\lim_{z\to 1}\left\{\frac{d^0}{dz^0}\left[\cancel{(z-1)}^1 . \frac{z.z^{n-1}}{\cancel{(z-1)}(z-2)}\right]\right\}$$

(1-1)

$$\text{or} \qquad \text{Residue} = \lim_{z\to 1}\left[\frac{z}{z-2}.z^{n-1}\right] = \lim_{z\to 1}\left[\frac{(z^1).z^n.z^{-1}}{z-2}\right] = \lim_{z\to 1}\left[\frac{z^n}{z-2}\right]$$

$$\text{or} \qquad \text{Residue} = -1 \qquad = \frac{1}{1-2} = \frac{1}{-1}$$

Similarly, for poles at z = 2

$$\text{Residue} = \frac{1}{0!}\lim_{z\to 2}\left\{\frac{d^0}{dz^0}\left[\cancel{(z-2)}^1 \frac{z.z^{n-1}}{(z-1)\cancel{(z-2)}}\right]\right\}$$

(4-1)

$$= \lim_{z\to 2}\left[\frac{z}{z-1}z^{n-1}\right] = 2.2^{n-1} = \left[\frac{z^1 . z^n . z^{-1}}{z-1}\right] = \frac{2^n}{2-1} = 2^n.$$

Residue = 2^n

Hence, $x(n) = \{-1 + 2^n\}.\, u[n]$

Example 3.37

Obtain the inverse z-transform of

$$X(n) = ln\,(1 + az^{-1}),\ |z| > |a|$$

Solution : According to logarithmic series expansion, we have

$$ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots$$

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Therefore, $X(z) = ln(1 + az^{-1})$

Simplifying, we have

$$X(z) = az^{-1} - \frac{1}{2}(az^{-1})^2 + \frac{1}{3}(az^{-1})^3 \dots$$

$$= az^{-1} - \frac{1}{2}a^2z^{-2} + \frac{1}{3}a^3z^{-3} \dots$$

Taking inverse z-transform, we obtain

$$x(n) = \left\{ \underset{\uparrow}{0}, a, -\frac{1}{2}a^2, \frac{1}{3}a^3, \right\}$$ Ans.

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Example 3.39

Find the inverse z-transform of

$$X(z) = \frac{z^3 + z^2}{(z-1)(z-3)} \qquad \text{ROC} : |z| > 3$$

Solution

$$\frac{X(z)}{z} = \frac{z^2 + z}{(z-1)(z-3)}$$; Here 2 poles are present, having order m = 1

Converting the above improper rational function ($\because$ M = N) into sum of a constant and a proper rational function we get

$$\frac{X(z)}{z} = 1 + \frac{5z-3}{(z-1)(z-3)}$$

The rational expression can be expanded by Partial fraction expansion

$$\frac{5z-3}{(z-1)(z-3)} = \frac{C_1}{z-1} + \frac{C_2}{z-3}$$

where

$$C_1 = (z-1)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=1} = -1$$

$$C_2 = (z-3)\frac{(5z-3)}{(z-1)(z-3)}\bigg|_{z=3} = 6$$

Therefore

$$\frac{X(z)}{z} = 1 - \frac{1}{z-1} + \frac{6}{z-3}$$

$$X(z) = z - \frac{z}{z-1} + \frac{6z}{z-3}$$

Taking Inverse z-transform on both sides we get

$$x(n) = \delta(n+1) - u(n) + 6(3)^n u(n)$$

Example 3.40

Use the residue method to find the inverse z-transform of

$$X(z) = \frac{z}{(z-2)(z-3)} \; |z| < 2$$

Solution :

In this case there are two poles z = 3 and z = 2 outside the ROC | z | < 2, so the sequence is non causal.

For n < 0

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Example 3.41

Find the inverse z-transform of $X(z) = \frac{z^2 + z}{(z-1)(z-3)}$, ROC : $|z| > 3$. Using (a) Partial fraction expansion method (b) Residue method (c) Convolution Method.

Solution

P-163

Ex:- 3.41

(b)

A) Here ROC is $|z| > 3$, so the sequence is causal signal. Because ROC is outward of the outermost pole.

Here two poles available $z = 1, 3$ having order $m = 1$.

Residue at pole $z = 1$ having order $m = 1$:-

$$R_1 = \lim_{z \to 1} \left[(z-1) \cdot X(z) \cdot z^{n-1} \right]$$

$$= \lim_{z \to 1} \left[(z-1) \cdot \frac{z^2+z}{(z-1)(z-3)} \cdot z^{n-1} \right] = \lim_{z \to 1} \left[\frac{z^1(z+1) \cdot z^n \cdot z^{-1}}{z - 3} \right]$$

$$= \lim_{z \to 1} \left[\frac{(1+1) \cdot (1)^n}{1-3} \right] = \frac{2 \cdot 1}{-2} = -1 .$$

Residue at Pole $z = 3$;

$$R_2 = \lim_{z \to 3} \left[(z-3) \cdot X(z) \cdot z^{n-1} \right]$$

$$= \lim_{z \to 3} \left[(z-3) \cdot \frac{z^2 + z}{(z-1)(z-3)} \cdot z^{n-1} \right]$$

$$= \lim_{z \to 3} \left[\frac{z^1 (z+1) \cdot z^n \cdot z^{-1}}{(z-1)} \right] = \frac{3+1 \cdot 3^n}{3-1} = \frac{4(3)^n}{2}$$

$$\therefore x(n) = R_1 + R_2 = \left[-1 + 2(3)^n \right] u(n)$$

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Example 3.45

Find the system function, H(z) and unit-sample response h(n) of the system whose difference equation is described as under :

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

where y(n) and x(n) are the output and input of the system, respectively.

Solution : The given difference equation is

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

Taking the z-transform of above difference equation, we get

$$y(z) = \frac{1}{2}z^{-1}Y(z) + 2X(z)$$

or $$Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = 2X(z)$$

Here $x(n)$ = unit impulse $\delta(n)$

or $$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}} =$$

$x(z) = Z[\delta(n)] = 1$

So o/p is impulse response $y(n) = h(n)$.

This system function has a pole at $z = \frac{1}{2}$ and zero at $z = 0$.

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$ = system function

Also,

$$h(n) = \text{Inverse z-transform of } H(z) = Z^{-1}\left[\frac{2}{1 - \frac{1}{2}z^{-1}}\right]$$

or, $$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

This is the unit-sample response of the system. Ans.

Example 3.49

Determine the causal signal x(n) having the z-transform

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

P-173 Ex :- 3.49.

A) By using Residue Method

$$x(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2} = \frac{1}{\left(1-\frac{2}{z}\right)\left[1-\frac{1}{z}\right]^2}$$

$$= \frac{1}{\left(\frac{z-2}{z}\right)\left(\frac{z-1}{z}\right)^2} = \frac{1}{\left(\frac{z-2}{z}\right)\cdot\left[\frac{(z-1)^2}{z^2}\right]} = \frac{1}{\frac{(z-2)(z-1)^2}{z^3}}$$

$$= \frac{z^3}{(z-2)(z-1)^2}$$; Here $x(z)$ is having two poles at $z=2$ having order 1, at $z=1$, having order $m=2$.

Residue at pole $z=2$:-

$$R_1 = \lim_{z\to 2}\left[(z-2)\cdot x(z)\cdot z^{n-1}\right] = \lim_{z\to 2}\left[z-2\cdot\frac{z^3}{(z-2)(z-1)^2}\cdot z^{n-1}\right]$$

$$= \lim_{z\to 2}\left[\frac{z^3\cdot z^n \cdot z^{-1}}{(z-1)^2}\right] = \lim_{z\to 2}\left[\frac{z^2\cdot z^n}{(z-1)^2}\right]$$

$$= \frac{2^2\cdot 2^n}{(2-1)^2} = 4\cdot 2^n$$

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Residue at pole $Z = 1$ having order $m = 2$:

$0! = 1$
$1! = 1$

$R_2 = \cancel{\lim_{Z \to 1} \left[Z \right.}$

$$R_2 = \frac{1}{(m-1)!} \lim_{Z \to 1} \frac{d^{m-1}}{dZ^{m-1}} \left[(Z-1)^m \cdot X(Z) \cdot Z^{n-1} \right]$$

$$\frac{dx^n}{dx} = n \cdot x^{n-1}$$

Here $m = 2$

$$= 1 \cdot \lim_{Z \to 1} \cdot \frac{d}{dZ} \left[(Z-1)^2 \cdot \frac{Z^3}{(Z-2)(Z-1)^2} \cdot Z^{n-1} \right]$$

$$= \lim_{Z \to 1} \frac{d}{dZ} \left[\frac{Z^3 \cdot Z^{-1} \cdot Z^n}{Z-2} \right] = \lim_{Z \to 1} \frac{d}{dZ} \left[\frac{Z^{n+2}}{Z-2} \right]$$

$$= \lim_{Z \to 1} \frac{\frac{dZ^{n+2}}{dZ} \cdot (Z-2) - Z^{n+2} \cdot \frac{d(Z-2)}{dZ}}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{(n+2) \cdot Z^{n+2-1} \cdot (Z-2) - Z^{n+2} \cdot 1}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{n+2 \cdot Z^{n+1} \cdot (Z-2) - Z^{n+2}}{(Z-2)^2}$$

$$= \lim_{Z \to 1} \frac{n+2 \cdot Z^n \cdot Z^1 \cdot (Z-2) - Z^n \cdot Z^2}{(Z-2)^2}$$

$$= \frac{n+2 \cdot 1^n \cdot 1^1 \cdot (1-2) - 1^n \cdot 1^2}{(1-2)^2} = \frac{-(n+2) - 1}{1}$$

$$= -n - 3 = -(n+3)$$

$$\therefore x(n) = R_1 + R_2$$

$$= \left[4(2)^n - (n+3) \right] u(n)$$

NOTE:- $Z\left[x(n-1)\right] = Z^{-1} X(Z) + x(-1)$

where $x(-1)$ represent the inital condition.

$$Z\left[x(n-2)\right] = Z^{-2} X(Z) + Z^{-1} x(-1) + x(-2)$$

$$z[x(n-3)] = z^{-3}\cdot X(z) + z^{-2}\cdot x(-1) + z^{-1}\cdot x(-2) + x(-3)$$

where $x(-1)$, $x(-2)$, $x(-3)$ are initial conditions.

Que. Solve the difference equation

$y(n) + 3y(n-1) = x(n)$, where $x(n)$ is $\left(\frac{1}{2}\right)^n\cdot u(n)$

& $y(-1) = 2$.

A) Apply z-transform on both sides to the given difference equation:

$$Y(z) + 3[z^{-1}\cdot Y(z) + y(-1)] = X(z)$$

$$\Rightarrow Y(z) + 3[z^{-1}Y(z) + 2] = X(z)$$

$$\Rightarrow Y(z) + 3z^{-1}\cdot Y(z) + 6 = X(z)$$

$$\Rightarrow Y(z)[1 + 3\cdot z^{-1}] = -6 + X(z)$$

$$\Rightarrow Y(z) = \frac{-6}{1+3z^{-1}} + \frac{X(z)}{(1+3z^{-1})} = \frac{-6}{1+3z^{-1}} + \frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$\Rightarrow Y(z) = \frac{-6}{1+3z^{-1}} + \frac{A}{1+3z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}}$$

$$A = \lim_{z^{-1}\to -1/3} (1+3z^{-1}) \frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{1}{1-\frac{1}{2}\times\left(-\frac{1}{3}\right)} = \frac{1}{1+1/6} = \frac{6}{7}$$

$$B = \lim_{z^{-1}\to 2} \left(1-\frac{1}{2}z^{-1}\right)\cdot\frac{1}{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$= \frac{1}{1+3\times 2} = \frac{1}{1+6} = 1/7$$

$$\therefore Y(z) = -6\cdot\frac{1}{1+3z^{-1}} + \frac{6}{7}\cdot\frac{1}{1+3z^{-1}} + \frac{1}{7}\cdot\frac{1}{1-\frac{1}{2}z^{-1}}$$

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Applying z-inverse transform on both sides we will get $\Rightarrow y(n) = -6 \cdot (-3)^n \cdot u(n) + \frac{6}{7}(-3)^n \cdot u(n) + \frac{1}{7}\left(\frac{1}{2}\right)^n \cdot u(n)$

$$\Rightarrow y(n) = \frac{-36}{7} \cdot (-3)^n \cdot u(n) + \frac{1}{7} \cdot \left(\frac{1}{2}\right)^n \cdot u(n)$$

$X(z)$

Example 3.54

Determine the impulse response and the step response of the following causal system. Determine it is stable or not.

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$$

Solution

$$y(n)=\frac{3}{4}y(n-1)-\frac{1}{8}y(n-2)+x(n)$$

Inpulse response means $x(n)=\sigma(n)$, here $y(n)=h(n)$

Applying Z - transform to equation (1),

$$Y(z)=\frac{3}{4}Z^{-1}Y(z)-\frac{1}{8}z^{-2}Y(z)+X(z)$$

$$\Rightarrow Y(z)\left[1-\frac{3}{4}z+\frac{2}{8}z^{-2}\right]=X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)}=\frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z}-2$$

$$\Rightarrow H(z)=\frac{8z^2}{8z^2-6z+1}$$

$$\Rightarrow \frac{H(z)}{z}=\frac{8z}{8z^2-6z+1}$$

$$=\frac{8z}{(2z-1)(4z-1)}$$

$$=\frac{A}{1z-1}+\frac{B}{4z-1}$$

$$A=\left.\frac{8z}{4z-1}\right|_{z=\frac{1}{2}}=4$$

$$B=\left.\frac{8z}{2z-1}\right|_{z=\frac{1}{4}}=-4$$

$$\frac{H(z)}{z}=\frac{4}{2z-1}-\frac{4}{4z-1}\Rightarrow 1+(z)=\frac{4z}{2z-1}-\frac{4z}{4z-1}$$

$$\Rightarrow h(n)=\left[2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\right]u(n)$$

For step response, $x(n)=u(n)$

$$\Rightarrow X(z)=\frac{Z}{z-1}$$

So Y(z) = H(z) X(z)

$$=\left[\frac{4z}{2z-1}-\frac{4z}{4z-1}\right]\cdot\frac{Z}{z-1}$$

$$=\frac{8z^2}{(z-1)(2z-1)(4z-1)}$$

$$\Rightarrow\frac{Y(z)}{z}=\frac{8z}{(z-1)(2z-1)(4z-1)}$$

$$=\frac{A}{z-1}+\frac{B}{2z-1}+\frac{C}{4z-1}$$

$$A=\left.\frac{8z}{(2z-1)(4z-1)}\right|_{Z=1}=\frac{8}{3}$$

$$B=\left.\frac{8z}{(z-1)(4z-1)}\right|_{Z=\frac{1}{2}}$$

$$=-8$$

$$=\frac{2}{\left(\frac{-3}{4}\right)\left(\frac{-1}{2}\right)}$$

$$=\frac{16}{3}$$

$$\text{So }Y(z)=\frac{8}{3}\frac{z}{z-1}-4\frac{2z}{2z-1}+\frac{4}{3}\frac{4z}{4z-1}$$

$$y(n)=\frac{8}{3}u(n)-4\left(\frac{1}{2}\right)^n u(n)+\frac{4}{3}\left(\frac{1}{4}\right)^n un$$

Example 3.55

We want to design a causal discrete-time LTI system with the property that if the input is.

$$x(n)=\left(\frac{1}{2}\right)^n u(n)-\frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

then the output is

$$y(n)=\left(\frac{1}{3}\right)^n u(n)$$

(a) Determine the impulse response h(n) and the system function H(z) of a system that satisfies the foregoing conditions.

(b) Find the difference equation that characterizes this system.

(c) Determine if the system is stable.

Solution :

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$y(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4} z^{-1} \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{2z}{2z-1} - \frac{1}{2}\frac{1}{2z-1}$$

$$= \frac{4z-1}{2(2z-1)}$$

$$y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{1}{1-\frac{1}{3z}}$$

$$Y(z) = \frac{3z}{3z-1}$$

(a) $$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{6z(2z-1)}{(3z-1)(4z-1)}$$

$$\Rightarrow \frac{H(z)}{Z} = \frac{6(2z-1)}{(3z-1)(4z-1)}$$

$$= \frac{A}{3z-1} + \frac{B}{4z-1}$$

$$A = \left.\frac{6(2z-1)}{4z-1}\right|_{z=\frac{1}{3}}$$

$$= \frac{6.\left(\frac{-1}{3}\right)}{\frac{4}{3}-1} = -6$$

$$B = \left.\frac{6(2z-1)}{3z-1}\right|_{z=\frac{1}{4}}$$

$$= \frac{6\left(\frac{-1}{2}\right)}{\frac{-1}{4\,2}} = 12$$

$$\boxed{H(z)\frac{-6z}{3z-1} + \frac{12z}{4z-1}} \qquad \text{Ans (a)}$$

$$\Rightarrow \boxed{h(n) = -2.\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{4}\right)^n u(n)} \qquad \text{Ans (a)}$$

(b) $H(z) = \frac{Y(z)}{X(z)} = \frac{6(z)(2z-1)}{(3z-1)(4z-1)}$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{12z^2 - 6z}{12z^2 - 7z + 1}$$

$$\Rightarrow 12z^2Y(z) - 7zY(z) + Y(z) = 12z^2X(z) - 6zX(z)$$

So difference equation is,

$12y(n+2) - 7\,y(n+1) + y(n) = 12x\,(n+2) - 6x\,(n+1)$

(c) $\sum_{n=-\infty}^{\infty} |h(n)|$

$$= \left|-2\frac{1}{1-\frac{1}{3}} + 3\frac{1}{1-\frac{1}{4}}\right|$$

$$= |-3+4| = 1 \leq \infty$$

so it is a stable system.

DISCRETE TIME FOURIER TRANSFORM (DTFT)

↳ If $x(n)$ is Discreete Time signal Then DTFT is given by

$$X(w) = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-jwn}$$

↳ In Time Domain The signal is Discreete. But in frequency Domain the signal is continuous and periodic over the range 2π.

↳ In DTFT The Time Domain signal is Discreete and Non periodic and the frequency Domain signal is continuous and periodic.

↳ similarly from $X(w)$ we can obtain Time domain signal $x(n)$ as:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w) \cdot e^{jwn} \cdot dw$$

↳ DTFT obeys periodicity property, such as

$$X(w + 2\pi K) = X(w)$$

where K is integer.

Que:- If $x(n) = a^n \cdot u(n)$, Then find DTFT of given signal.

Soln:- $x(n) = a^n$; for $n \geq 0$ [Because $u(n) = 1$; for $n \geq 0$]

$= 0$; $n < 0$

$$X(w) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-jwn} = \sum_{n=0}^{\infty} a^n \cdot e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \underbrace{(a \cdot e^{-jw})}_{x}{}^n = 1 + a \cdot e^{-jw} + (a \cdot e^{-jw})^2 + (a \cdot e^{-jw})^3 + \cdots$$

$$= \boxed{\frac{1}{1 - a \cdot e^{-jw}}}$$

$\left[\text{since } \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}\right]$

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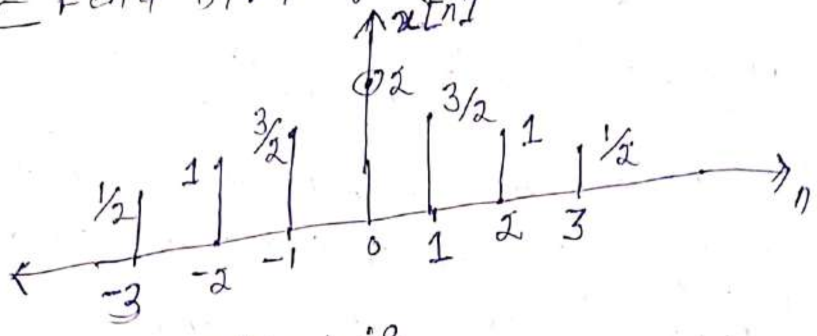
similarly $(-a)^n u(n) \xleftrightarrow{DTFT} \dfrac{1}{1+a\cdot e^{-j\omega}}$

— x —

Que:- Find DTFT of given signal $x[n]$



Soln:- Given signal is

$$x[n] = \left\{ \underset{\substack{\uparrow \\ n=-3}}{\frac{1}{2}}, \underset{\substack{\uparrow \\ n=-2}}{1}, \underset{\substack{\uparrow \\ n=-1}}{\frac{3}{2}}, \underset{\substack{\uparrow \\ n=0}}{2}, \underset{\substack{\uparrow \\ n=1}}{\frac{3}{2}}, \underset{\substack{\uparrow \\ n=2}}{1}, \underset{\substack{\uparrow \\ n=3}}{\frac{1}{2}} \right\}$$

$$X(\omega) = \sum_{n=-3}^{3} x[n]\cdot e^{-j\omega n}$$

$$= x(-3)\cdot e^{-j\omega(-3)} + x(-2)\cdot e^{-j\omega(-2)} + x(-1)\cdot e^{-j\omega(-1)} + x(0)\cdot e^{-j\omega(0)} + x(1)\cdot e^{-j\omega(1)} + x(2)\cdot e^{-j\omega(2)} + x(3)\cdot e^{-j\omega(3)}$$

$$= \frac{1}{2}\cdot e^{j\omega 3} + 1\cdot e^{j\omega 2} + \frac{3}{2}\cdot e^{j\omega} + 2\cdot e^{0} + \frac{3}{2}\cdot e^{-j\omega} + 1\cdot e^{-j2\omega} + \frac{1}{2}\cdot e^{-j\omega 3}$$

$$= \frac{1}{2}\left[e^{-j\omega 3} + e^{j\omega 3}\right] + 1\cdot\left[e^{j\omega 2} + e^{-j2\omega}\right] + \frac{3}{2}\left[e^{-j\omega} + e^{j\omega}\right] + 2$$

$$= \frac{1}{2}\times 2\left[\frac{e^{j3\omega} + e^{-j3\omega}}{2}\right] + 2\left[\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right] + \frac{3}{2}\cdot 2\left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right] + 2$$

$$= \cos 3\omega + 2\cdot\cos 2\omega + 3\cdot\cos\omega + 2$$

Here signal is Even symmetry.

$\left[\cos\theta = e^{j\theta} + e^{-j\theta}\right]$

Que:- The signal $x[n] = \left(\frac{1}{2}\right)^n \cdot u(n)$, $y(n) = x^2(n)$
Find DTFT for the signal $y(n)$ i.e. $Y(e^{jw})$
and $Y(e^{j\cdot 0})$

Sol^n:- Given that $x(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$,

$$y(n) = x^2(n) = \left[\left(\frac{1}{2}\right)^n \cdot u(n)\right]^2 = \left(\frac{1}{2}\right)^{2\cdot n} \cdot u(n)$$

$$= \left(\frac{1}{4}\right)^n \cdot u(n)$$

Now DTFT for the signal $y(n)$ is

$$Y(w) = Y(e^{jw}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot u(n) \cdot e^{-jwn} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot 1 \cdot e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4} \cdot e^{-jw}\right)^n$$

$$= \left(\frac{1}{4} \cdot e^{-jw}\right)^0 + \left(\frac{1}{4} \cdot e^{-jw}\right)^1 + \left(\frac{1}{4} \cdot e^{-jw}\right)^2 + \left(\frac{1}{4} \cdot e^{-jw}\right)^3 + \ldots$$

$$= \frac{1}{1 - \frac{1}{4} e^{-jw}} \qquad \left[\because \sum_{n=0}^{\infty} a^n = a^0 + a^1 + a^2 + \ldots = \frac{1}{1-a}\right.$$

Take $a = \frac{1}{4} \cdot e^{-jw}$

$$Y(0) = Y(e^{j\cdot 0}) = \frac{1}{1 - \frac{1}{4} \cdot e^{-j\times 0}} = \frac{1}{1 - \frac{1}{4} \times 1} \quad \left[e^{-0} = 1\right]$$

$$= \frac{1}{\frac{4-1}{4}} = \frac{4}{3}$$

$Y(0)$ = Spectrum at origin

$$Y(w) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-jwn} \Rightarrow Y(0) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-j\cdot 0 \cdot n}$$

$$\Rightarrow Y(0) = \sum_{n=-\infty}^{\infty} y[n]$$

$\Rightarrow$ Frequency Domain signal at origin = sum of Time Domain signals = Area under Time Domain signal.

—— x ——

DFT (Discrete Fourier Transform) :-

↳ Here Time Domain signal is periodic, Discrete and Frequency Domain signal is also Discrete and periodic.

Discrete Fourier Transform [D.F.T.] :- (1)

$$x[n] \xleftrightarrow[\text{DFT}]{N\text{-point}} X(K) \quad \text{--- (1)}$$

$$X(K) = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-j2\pi Kn}{N}}, \quad K = 0, 1, 2, \cdots N-1$$

$$\text{--- (2)}$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot e^{\frac{j2\pi Kn}{N}}, \quad n = 0, 1, 2, \cdots N-1$$

$$\omega_N = e^{-j2\pi/N} = \text{phase factor, Twiddle Factor}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot \omega_N^{Kn}, \quad x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot \omega_N^{-Kn}$$

↳ In Discrete Time Fourier Transform (D.T.F.T.)

$$x[n] \xleftrightarrow{\text{D.T.F.T.}} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

continuous frequency, Discrete Frequency

$$\omega_d = \frac{\text{Analog Frequency } (\omega_a)}{T_s}$$

where, T_s = Sampling Interval

By putting $K=0$ in eq^n-(1), $X(0) = \sum_{n=0}^{N-1} x[n]$

$$\Rightarrow X(0) = x[0] + x[1] + x[2] + \cdots + x[N-1] \quad \text{--- (3)}$$

By putting $K = N/2$ if N is even number in eq^n-(1)

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n) \cdot (-1)^n \quad \text{--- (4)}$$

$$= x(0) - x[1] + x[2] - x[3] + x[4] \cdots$$

$$e^{\frac{-j2\pi}{N} \cdot \frac{N}{2} \cdot n} = e^{-j\pi n} = (-1)^n, \quad e^{-j\pi} = -1$$

By Adding Equation-(3) and Equation-(4)

$$X(0) + X\left(\frac{N}{2}\right) = 2\left[x(0) + x(2) + x(4) + \cdots\right]$$

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By subtracting Equation-④ from Equation-③

$$X(0)-X\left(\frac{N}{2}\right)=2\left[x(1)+x(3)+x(5)+\cdots\cdots\right]$$

$$X(K)=\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ \vdots\\ X(N-1)\end{bmatrix}=X_N$$

$$\left[W_N\right]=\begin{array}{c|ccccc} & n=0 & n=1 & n=2 & \cdots & n=N-1\\ \hline K=0 & W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0\\ K=1 & W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1}\\ K=2 & W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)}\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ K=N-1 & W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2}\end{array}$$

$$x(n)=\begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ \vdots\\ x(N-1)\end{bmatrix}=x_N$$

$$=\left[W_N^{Kn}\right]$$ in Matrix format

In vector form we can write

$$X_K=[W_N]\cdot x_N \quad \text{and} \quad x_N=\frac{1}{N}\cdot[W_N]^*\cdot X_K$$

$$X[K]=\sum_{n=0}^{N-1}x[n]\cdot W_N^{Kn},\quad x[n]=\frac{1}{N}\sum_{K=0}^{N-1}X(K)\cdot \underbrace{W_N^{-Kn}}_{(W_N^*)^{Kn}}$$

For any 1 value of K number of computational multiplications $=N$, and number of Additions $=(N-1)$, so N numbers of values for 'K' the Total Number of Multiplications require $=N^2\,(N\cdot N)$ and Total Number of Additions require $=(N-1)\cdot N=N^2-N$

↳ $x(t)\xleftrightarrow{\text{Fourier Transform}}X(\omega)$

$$X(\omega)=\int_{-\infty}^{\infty}x(t)\cdot e^{-j\omega_a t}\cdot dt$$ where ω_a = Analog Frequency

$t\xrightarrow{\text{By sampling}}n\cdot T_s$

$x(t)\longrightarrow x[n]$

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2

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega_a \cdot n \cdot T_s}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega_d \cdot n}$$

$\boxed{\omega_d = \omega_a \cdot T_s}$, $2\pi f_d = 2\pi f_a \cdot \frac{1}{f_s}$

$\Rightarrow \boxed{f_d = \frac{f_a}{f_s}}$

f_d = Digital Frequency
f_a = Analog Frequency,
f_s = sampling Frequency.

properties of Twiddle Factor :-

$$\omega_N = e^{-j2\pi/N}$$

(1) $\omega_N^{K+N} = \omega_N^K$,

(2) $\omega_N^{K+\frac{N}{2}} = -\omega_N^K$

(3) $W_{N/2} = W_N^2$

PROPERTIES OF DFT :-

$$X(K) = \sum_{n=0}^{N-1} x[n] \cdot W_N^{Kn}, \quad K = 0, 1, 2, \ldots N-1$$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot W_N^{-Kn}, \quad n = 0, 1, 2, \ldots N-1$$

(1) periodicity property :-

If $x(n+N) = x(n)$ Then $X(K+N) = X(K)$

(2) Linearity property :-

If $x_1[n] \xleftrightarrow{DFT} X_1(K)$
$x_2[n] \xleftrightarrow{DFT} X_2(K)$

$$\alpha \cdot x_1[n] + \beta \cdot x_2[n] \xleftrightarrow{DFT} \alpha \cdot X_1(K) + \beta \cdot X_2(K)$$

(3) Time shifting property :-

If $x[n] \xleftrightarrow{DFT} X(K)$

Then, $x((n-n_0))_N \xleftrightarrow{DFT} X(K) \cdot e^{-j\frac{2\pi}{N}Kn_0}$

$X(K) \cdot \omega_N^{Kn_0}$

$x((n+n_0))_N \xleftrightarrow{DFT} X(K) \cdot e^{j\frac{2\pi}{N}Kn_0}$

$\approx X(K) \cdot W_N^{-Kn_0}$

(4) Frequency shifting property :-

If $x[n] \xleftrightarrow{DFT} X(K)$ Then

$x[n] \cdot e^{j\frac{2\pi \cdot l}{N} \cdot n} \xleftrightarrow{DFT} X((K-l))_N$

(5) Expansion in Time property :-

In DTFT, $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$, periodic with 2π

$x[\frac{n}{K}] \xleftrightarrow{DTFT} X(e^{j\omega K})$, periodic with $\frac{2\pi}{K}$

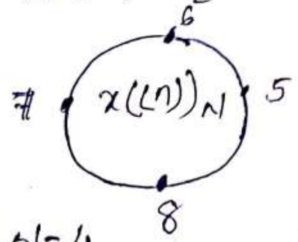
In DFT, $x[n] \longleftrightarrow X(K)$

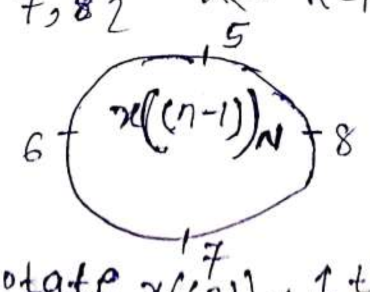
$\{x(0), x(1), x(2), \cdots x(N-1)\} \xleftrightarrow{DFT} \{X(0), X(1), X(2), \cdots X(N-1)\}$

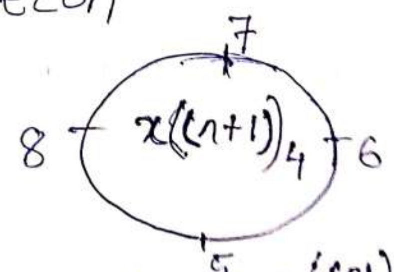
$x[\frac{n}{K}] \xleftrightarrow{DFT} \{\underbrace{X(0), X(1), X(2) \cdots X(N-1)}_{1st\ Repetition}, \underbrace{X(0), X(1), \cdots X(N-1)}_{2nd\ Repetition}$

$\underbrace{X(0), X(1), X(2), \cdots X(N-1)}_{Kth\ Repetition}\}$

$x(n) = \{5, 6, 7, 8\}$



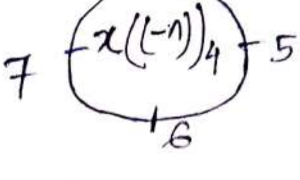




$N=4$

Rotate $x((n))_N$ 1 time in Anticlockwise Direction

Rotate $x((n))_4$ clockwise by 1 unit.



6. circulare convolution :-

$x_1[n] \xleftrightarrow{DFT} X_1(K)$, $x_2[n] \xleftrightarrow{DFT} X_2(K)$

$$w(n) = x_1(n) \; N \; x_2(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N$$

circulare convolution

$x_1(n)$, $x_2(n)$ sequence having equal length l. otherwise zeropadding will occure in the sequence, which has less number of samples.

$$w(n) = x_1(n) \; N \; x_2(n) \xleftrightarrow{DFT} X_1(K) \cdot X_2(K)$$

7. Multiplication property :-

$$x_1[n] \cdot x_2[n] \xleftrightarrow{DFT} \frac{1}{N} \cdot \left[X_1(K) \; N \; X_2(K) \right]$$

Que:- Find circulare convolution between Two sequence $x_1(n) = \{2, 3, 4, 5\}$, $x_2(n) = \{5, 6, 2, 1\}$

Soln:- $y(n) = x_1(n) \; N \; x_2(n)$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 2 & 5 & 4 & 3 \\ 3 & 2 & 5 & 4 \\ 4 & 3 & 2 & 5 \\ 5 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+30+8+3 \\ 15+12+10+4 \\ 20+18+4+5 \\ 25+24+6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 51 \\ 41 \\ 47 \\ 57 \end{bmatrix}$$

— x — x —

Que:- N-point DFT of $x[n] = a^n$, $0 \leq n \leq N-1$ is $X(K)$. Then the value of $X(K)$ for $K=2$, $a = 0.5$ and $N = 4$ is ——.

Soln:- $$X(K) = \sum_{n=0}^{N-1} x[n] \cdot e^{\frac{-j2\pi Kn}{N}} = \sum_{n=0}^{N-1} a^n \cdot \left(e^{\frac{-j2\pi K}{N}}\right)^n$$

$$= \sum_{n=0}^{N-1} \left(a \cdot e^{\frac{-j2\pi K}{N}}\right)^n$$

we know, $\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$

By applying this to above Equation, we will get

$$= \frac{1-\left(a\cdot e^{-j2\pi K/N}\right)^N}{1-a\cdot e^{-j2\pi K/N}}$$

$$\Rightarrow X(K) = \frac{1-(a)^N}{1-a\cdot e^{-j2\pi K/N}}$$

putting $K=2$,

$$\Rightarrow X(2) = \frac{1-(0.5)^4}{1-0.5\, e^{-j2\pi\times 2/4}}$$

$$\Rightarrow X(2) = \frac{1-(0.5)^4}{1-0.5\, e^{-j\pi}}, \quad \boxed{e^{-j\pi} = -1}$$

$$= \frac{1-0.0625}{1-0.5\times(-1)} = 0.625$$

x — x — x — x —

(8). CIRCULAR REVERSAL :-

If $x[n] \xleftrightarrow{DFT} X(K)$ Then

$$\boxed{x((-n))_N \xleftrightarrow{DFT} X((-K))_N}$$

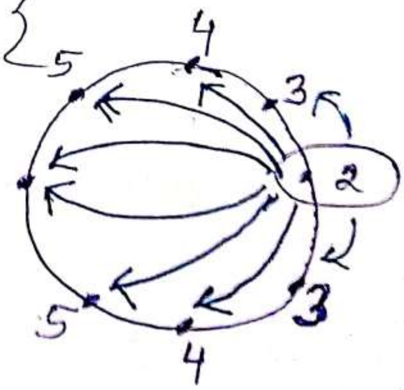
$$\boxed{x((-n))_N = x(N-n)}$$

(9.) circularly Even sequence :-

$x(N-n) = x(n)$, $1 \le n \le N-1$

The sequence will be symmetrical about origin point. $x[n] = \{2, 3, 4, 5, 6, 5, 4, 3\}$

↳ It is circularly even sequence



(10) circularly odd sequence :-

$x((-n))_N = x(N-n) = -x(n)$; $1 \le n \le N-1$.

↳ The sequence will be antisymmetric about the origin.

$x[n] = \{2, 3, 4, 5, 6, -5, -4, -3\}$

↑

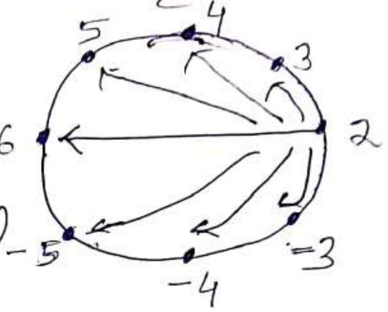
Not circularly odd sequence.

To be circularly odd sequence insted of 6, there should be 0.

$\underline{\underline{0}}$

$6+(-6)$

$= \underline{\underline{0}}$



(11) conjugate property :-

If $x[n] \xleftrightarrow{DFT} X(K)$ Then

$x^*[n] \xleftrightarrow{DFT} X^*((-K))_N$

(12) parseval's Relation :-

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \cdot \sum_{K=0}^{N-1} |X(K)|^2$$

(13) symmetry property of a 'real valued sequence :-

$x[n] \xleftrightarrow{DFT} X(K)$, Then

$$\boxed{X(N-K) = X^*(K)}$$

$$\boxed{\Rightarrow X(K) = X^*(N-K)}$$

Que:- If $X(K) = \{\underset{X(0)}{5}, \underset{X(1)}{2+j}, \underset{X(2)}{0}, \underset{X(3)}{0}, \underset{X(4)}{3+j}, -, -, -\}$

Then find the value of $x(0)$ is —.

Soln :- $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X[K] \cdot W_N^{-Kn}$, $x(0) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)$

$X(5) = X^*(8-5) = X^*(3) = \boxed{0}$, $X(6) = X^*(8-6) = X^*(2) = \boxed{0}$,

$X(7) = X^*(8-7) = X^*(1) = \boxed{2-j}$

$x(0) = \frac{1}{8}\left[X(0)+X(1)+X(2)+X(3)+X(4)+X(5)+X(6)+X(7)\right]$

By putting all these values we will get $x(0)$

Que:- Two sequence $[\overline{a,b,c}]$ and $[\overline{A,B,C}]$ are related as

$$\begin{bmatrix}A\\B\\C\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & W_3^{-1} & W_3^{-2}\\ 1 & W_3^{-2} & W_3^{-4}\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}, \text{ where } W_3 = e^{j\frac{2\pi}{3}}$$

If another sequence $[p,q,r]$ is derived as

$$\begin{bmatrix}p\\q\\r\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & W_3^{1} & W_3^{2}\\ 1 & W_3^{2} & W_3^{4}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & W_3^{2} & 0\\ 0 & 0 & W_3^{4}\end{bmatrix}\begin{bmatrix}A/3\\B/3\\C/3\end{bmatrix}$$

Then the relation between sequences $[p,q,r]$ and $[a,b,c]$ is

(A) $[p,q,r]=[b,a,c]$, (B) $[p,q,r]=[b,c,a]$

(C) $[p,q,r]=[c,a,b]$, (D) $[p,q,r]=[c,b,a]$

Soln:- $\sum_{n=-\infty}^{-1} a^{-n}\cdot e^{\frac{-j2\pi kn}{N}}$, $S=\sum_{n=-\infty}^{-1}\left(a\cdot e^{\frac{j2\pi k}{N}}\right)^{-n}$

Let $-n=m$, when $n\to-\infty$, $m\to+\infty$

$n\to-1$, $m\to+1$

$$S=\sum_{m=1}^{\infty}\Big(\underbrace{a\cdot e^{\frac{j2\pi k}{N}}}_{\text{Take } A}\Big)^{m} = A+A^2+A^3+\cdots$$

$$= A[1+A+A^2+A^3+\cdots]$$

$$= A\cdot\sum_{n=0}^{\infty}(A)^n = A\cdot\frac{1}{1-A}$$

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & W_N^{1} & W_N^{2}\\ 1 & W_N^{2} & W_N^{4}\end{bmatrix}\begin{bmatrix}x(0)\\x(1)\\x(2)\end{bmatrix}, \text{ if } W_N = e^{-j2\pi/N} \text{ for } N=3 \text{ here}$$

$$\underbrace{\begin{bmatrix}p\\q\\r\end{bmatrix}}_{x'(n)} = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & W_3^{1} & W_3^{2}\\ 1 & W_3^{2} & W_3^{4}\end{bmatrix}\underbrace{\begin{bmatrix}A\\W_3^{2}\cdot B\\W_3^{4}\cdot C\end{bmatrix}}_{X'(K)}$$

$x[n]=[a,b,c]$

$\downarrow$

$X(K)=[A,B,C]$

$x'[n] = [p, q, r]$

$X'[K] = [A, W_3^2 \cdot B, W_3^4 \cdot C]$

$A \rightarrow W_3^0 \cdot A$ we can consider

$X[K] = [A, B, C]$

$\downarrow$ $X(0), X(1), X(2)$

we can write, $X'[K] = W_N^{2K} \cdot X(K)$

$N=3$ it is given

$X'[K] = e^{j\frac{2\pi}{3}(2)K} \cdot X(K)$ $\rightarrow 2 = l$

we know, if $x[n] \xleftrightarrow{DFT} X(K)$ Then,

$x((n+l))_N \xleftrightarrow{DFT} e^{j\frac{2\pi K}{N} \cdot l} \cdot X(K)$

By comparing $l=2$ we will get.

$x[n] = [a, b, c] \longleftrightarrow X(K)$

$x((n+2))_3 \longleftrightarrow e^{j\frac{2\pi K}{N} \cdot 2} \cdot X(K) = X'[K]$

$\approx x'[n]$

Therefore $\boxed{x'[n] = x((n+2))_3}$

$[p, q, r] = [c, a, b]$

x —— x —— x —— x

FAST FOURIER TRANSFORM (FFT) :-

For calculating N-point DFT of $x(n)$,

$$X(K) = \sum_{n=0}^{N-1} x[n] \cdot W_N^{Kn} \quad , \text{ where } W_N = \text{Twiddle Factor} = e^{\frac{-j2\pi}{N}}$$

$$X[K] = x[0] \cdot W_N^{0 \cdot K} + x[1] \cdot W_N^{1 \cdot K} + x[2] \cdot W_N^{2 \cdot K} + \dots + x[N-1] \cdot W_N^{(N-1)K}$$

$K = 0, 1, 2, \dots N-1$

To calculate N-point DFT we require

(i) Total Number of complex Multiplications $= N \times N$
$= \boxed{N^2}$

(ii) Total Number of complex Additions $= N \cdot (N-1)$
$= N^2 - N$

RELATION BETWEEN DFT SEQUENCE AND FOURIER SERIES COEFFICIENTS :-

The periodic sequence $x_p(n)$ with period N is

$$x_p(n) = \sum_{K=0}^{N-1} c_K \cdot e^{j2\pi nK/N}, \quad \text{where } -\infty < n < \infty$$

where c_K is Fourier series coefficients.

$$c_K = \frac{1}{N} \cdot \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j2\pi Kn/N}, \quad \text{where } K = 0, 1, \cdots N-1 \quad \text{—— (1)}$$

If $x(n) \xleftrightarrow{DFT} X(K)$, By making Inverse DFT

$$x(n) = \frac{1}{N} \cdot \sum_{K=0}^{N-1} X(K) \cdot e^{j2\pi nK/N} \quad \text{—— (2)}$$

where $x(n)$ is Aperiodic sequence.

$x(n) = x_p(n)$ in $0 \leq n \leq N-1$

By comparing Equation-(1) and Equation-(2) we will get, $\boxed{c_K = \frac{X(K)}{N}}$

(or) $\boxed{X(K) = N \cdot c_K}$

—— x —— x ——

RELATION OF DFT WITH Z-TRANSFORM :-

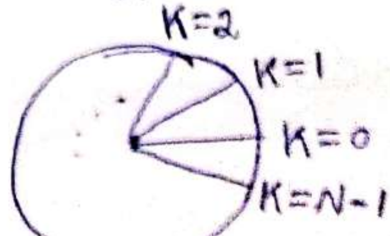
(1) DFT From Z-Transform

If $x(n)$ is sequence Then it's Z-Transform is

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

If $X(Z)$ is sampled on the unit circle and the sampling rate is uniform and the number of samples $= N$

Here radius $r = 1$



$$Z_K = e^{\frac{j2\pi K}{N}}$$

$$\text{DFT } X(K) = X(Z)\Big|_{\text{at } Z = e^{\frac{j2\pi K}{N}}}$$

$$X(K) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{\frac{j2\pi Kn}{N}}$$

$x(n)$ is limited to $n = 0, 1, 2, \cdots N$

$$\therefore X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nK}{N}}$$

$X(Z)$ to DFT, $X(Z) = \sum_{n=0}^{N-1} x(n) \cdot Z^{-n}$, where $n = 0, 1, \cdots N-1$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot e^{j2\pi Kn/N} \cdot Z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot \sum_{n=0}^{N-1} e^{j2\pi Kn/N} \cdot Z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi K}{N}} \cdot Z^{-1}\right)^n$$

$$= \frac{1}{N} \cdot \sum_{K=0}^{N-1} X(K) \cdot \frac{1 - \left(e^{j2\pi K/N} \cdot Z^{-1}\right)^N}{1 - e^{j2\pi K/N} \cdot Z^{-1}} \qquad \left[\because \sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a}\right]$$

$$\left(e^{j2\pi K/N} \cdot Z^{-1}\right)^N = e^{\frac{j2\pi KN}{N}} \cdot Z^{-N} = e^{j2\pi K} \cdot Z^{-N}, \quad e^{j2\pi K} = 1$$

$$\boxed{X(Z) = \frac{1 - Z^{-N}}{N} \cdot \sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j2\pi K/N} \cdot Z^{-1}}}$$

This is the relationship between DFT and Z-Transform.

— x — x —

FILTERING OF LONG DATA SEQUENCES :- (7)

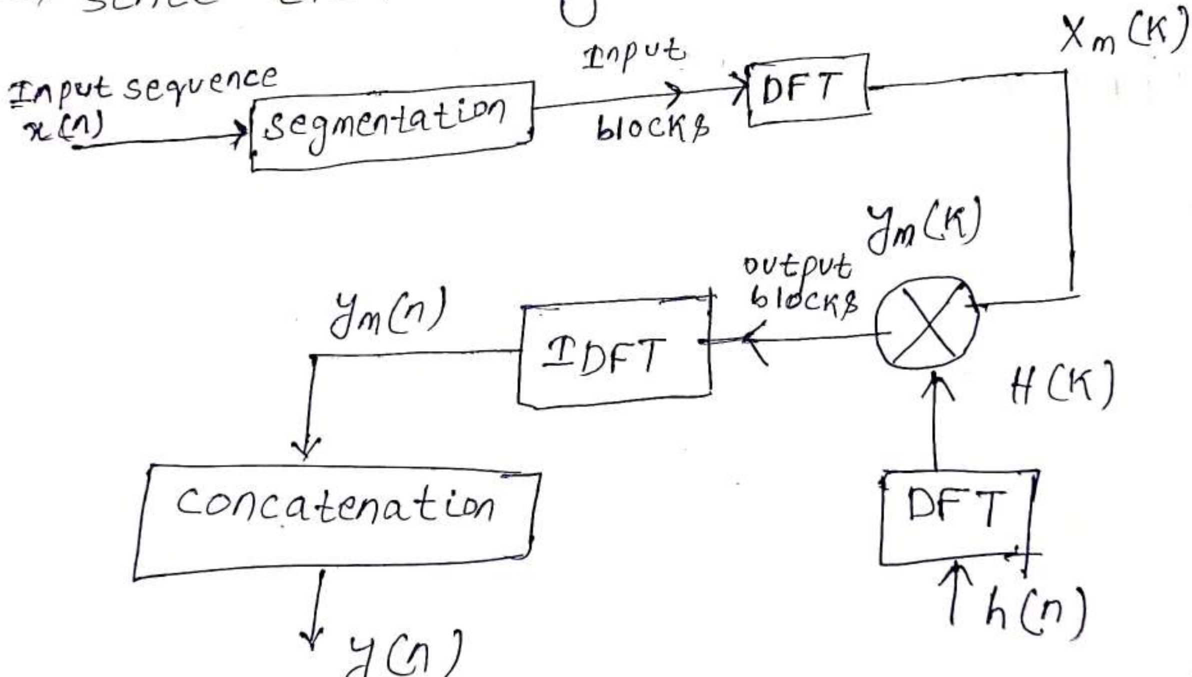
It can be very long

example :- Real Time signal processing

↳ Linear filtering using DFT must be done on a block of input data.

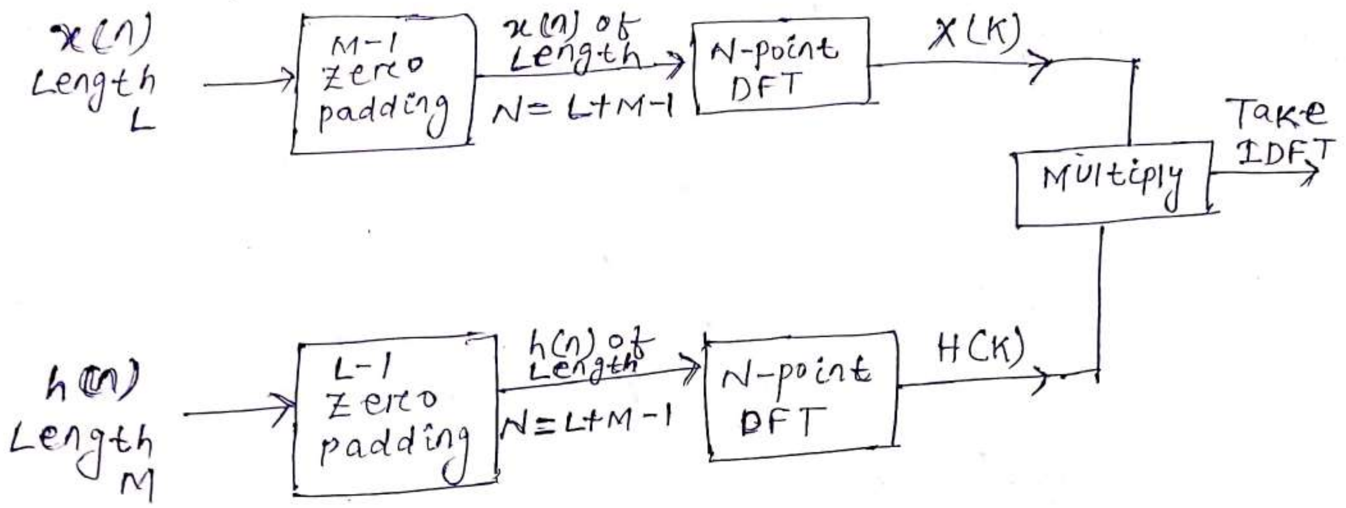
↳ At first Long data is segmented into Fixed size blocks.

↳ since the Filtering is a Linear process



↳ For segmentation, concatenation there are Two Methods available:
① overlap save method, ② overlap Add method.

DFT for Linear Filtering :-



$y(n) = x(n) * h(n)$

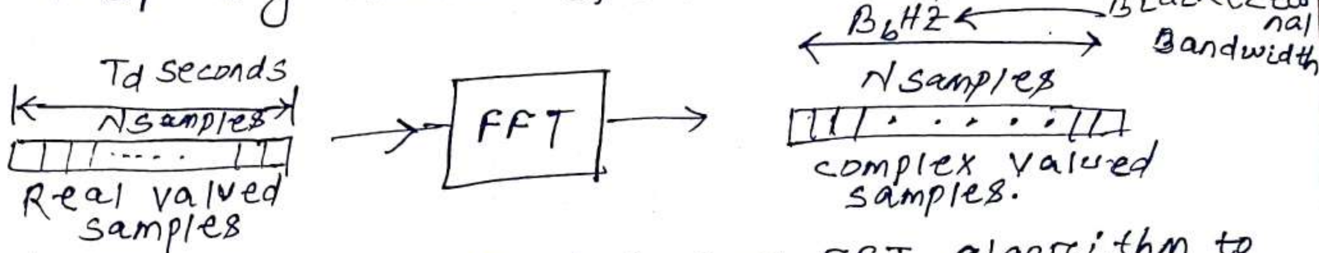
IDFT → Inverse Discrete Fourier Transform

— x — x — x — x — x —

FAST FOURIER TRANSFORM (FFT) BASICS :-

↳ It is fast computation algorithm for Discrete Fourier Transform (DFT).

↳ In FFT we are taking array of Time Domain waveform samples and producing array of Frequency domain spectrum samples.



↳ N must be a power of 2 for FFT algorithm to be truly "fast".

↳ In Input side Real valued samples we are giving, in output side complex valued samples we will get. Typically work with magnitude and phase Representation of the complex values.

↳ Sampling Interval $\Delta t = \frac{T_d}{N}$, sampling Frequency $\frac{1}{\Delta t} = \frac{N}{T_d} = f_s\,(Hz)$
In time Domain

Similarly In frequency Domain, $\Delta f = \frac{B_b}{N} = \frac{f_s}{N}$ = Frequency spacing

$f_{max} = \frac{B_b}{2} = \frac{f_s}{2}$ ⇒ Typically display only Lower Half of the output Array.

Nyquist frequency

↳ FFT is an efficient way (or) Algorithm to compute DFT with reduced ~~com-~~ computations. It is not a Transform. It is an algorithm.

Radix-2 FFT Algorithms :-

DIT
(Decimation in Time)

DIF
(Decimation in Frequency)

Both Algorithms use Divide and conquer Approach.

↳ we have to choose the signal length 'N' such that it can be factored like

$$N = r_1 \cdot r_2 \cdot r_3 \cdot \cdots \cdot r_m$$

If $r_1 = r_2 = \cdots = r_m = r$

Then $\boxed{N = r^m}$

where $r \rightarrow$ Represents Radix.

For Radix-2, $r = 2$, $N = 2^m$

N-point DFT

Divides into

$\frac{N}{2}$ point DFT | $\frac{N}{2}$ point DFT

$\frac{N}{4}$ point DFTs | $\frac{N}{4}$ point DFTs | $\frac{N}{4}$ point DFTs | $\frac{N}{4}$ point DFTs

It will continue the division array like this until 2-point DFTs we obtaine.

① symmetry property: $W_N^{K+\frac{N}{2}} = -W_N^K$

proof:- $W_N^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(K+\frac{N}{2})} = e^{-j\frac{2\pi}{N}\cdot K} \cdot e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}}$

$= e^{-j\frac{2\pi}{N}\cdot K} \cdot e^{-j\pi} = \boxed{-W_N^K}$

② periodicity:- $W_N^{K+N} = W_N^K$

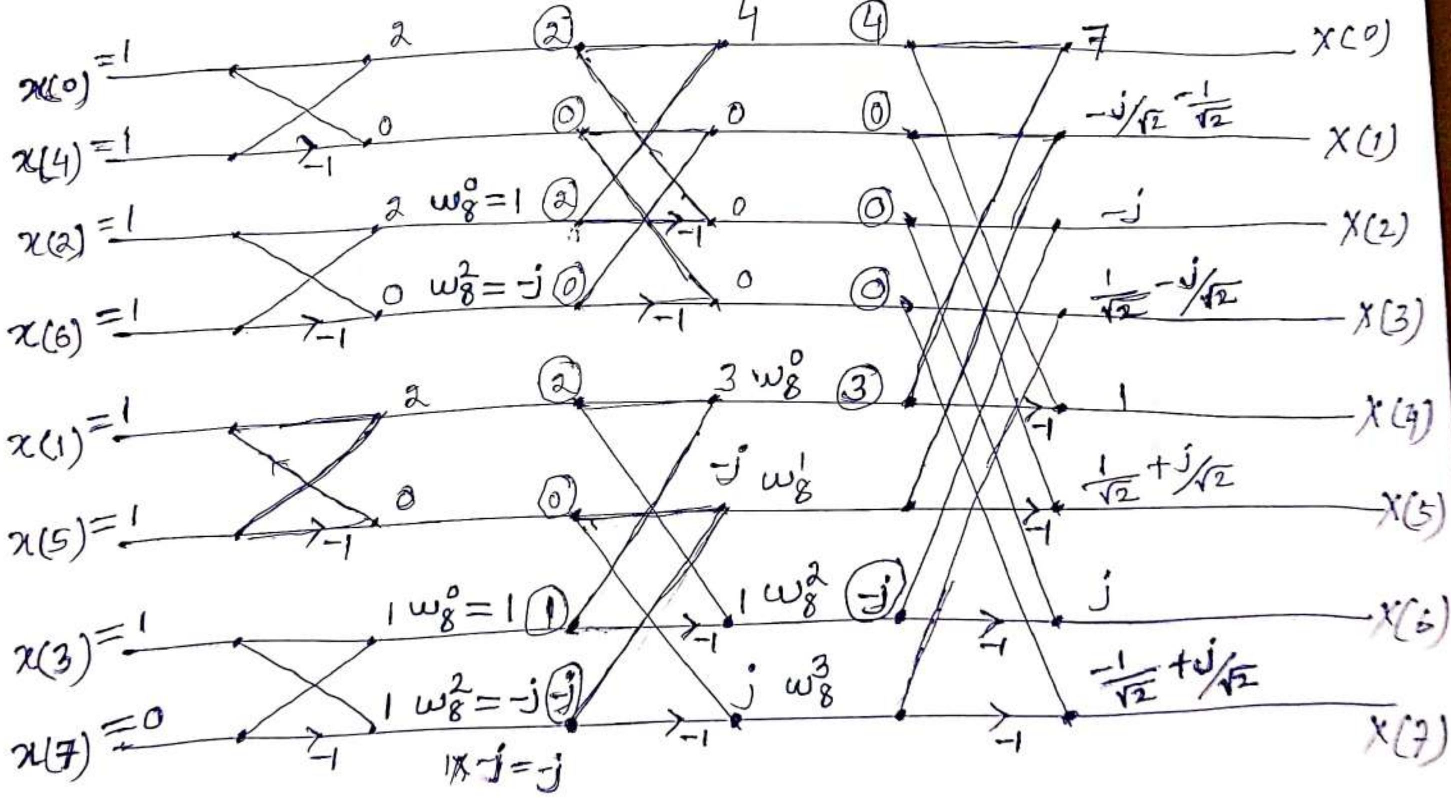
③ $W_N^2 = e^{-j\frac{2\pi}{N}\cdot(2)}$

$= e^{\frac{-j2\pi}{(N/2)}} = W_{N/2}$

DIT-FFT [DECIMATION IN TIME]

Que:- Find the 8 point DFT of $x(n)$ $= \{1,1,1,1,1,1,1,0\}$ using radix-2 DIT-FFT algorithm. (or) compute the DFT for the sequence $\{1,1,1,1,1,1,1,0\}$ using DIT-FFT algorithm.

Soln:-



Here $W_8^0=1$, $W_8^1=\frac{1}{\sqrt{2}}-j/\sqrt{2}$, $W_8^2=-j$, $W_8^3=-\frac{1}{\sqrt{2}}-j/\sqrt{2}$

$3\times W_8^0=3\times1=3$, $-j\times W_8^1=-j\left(\frac{1}{\sqrt{2}}-j/\sqrt{2}\right)=\boxed{\frac{-j}{\sqrt{2}}-\frac{1}{\sqrt{2}}}$,

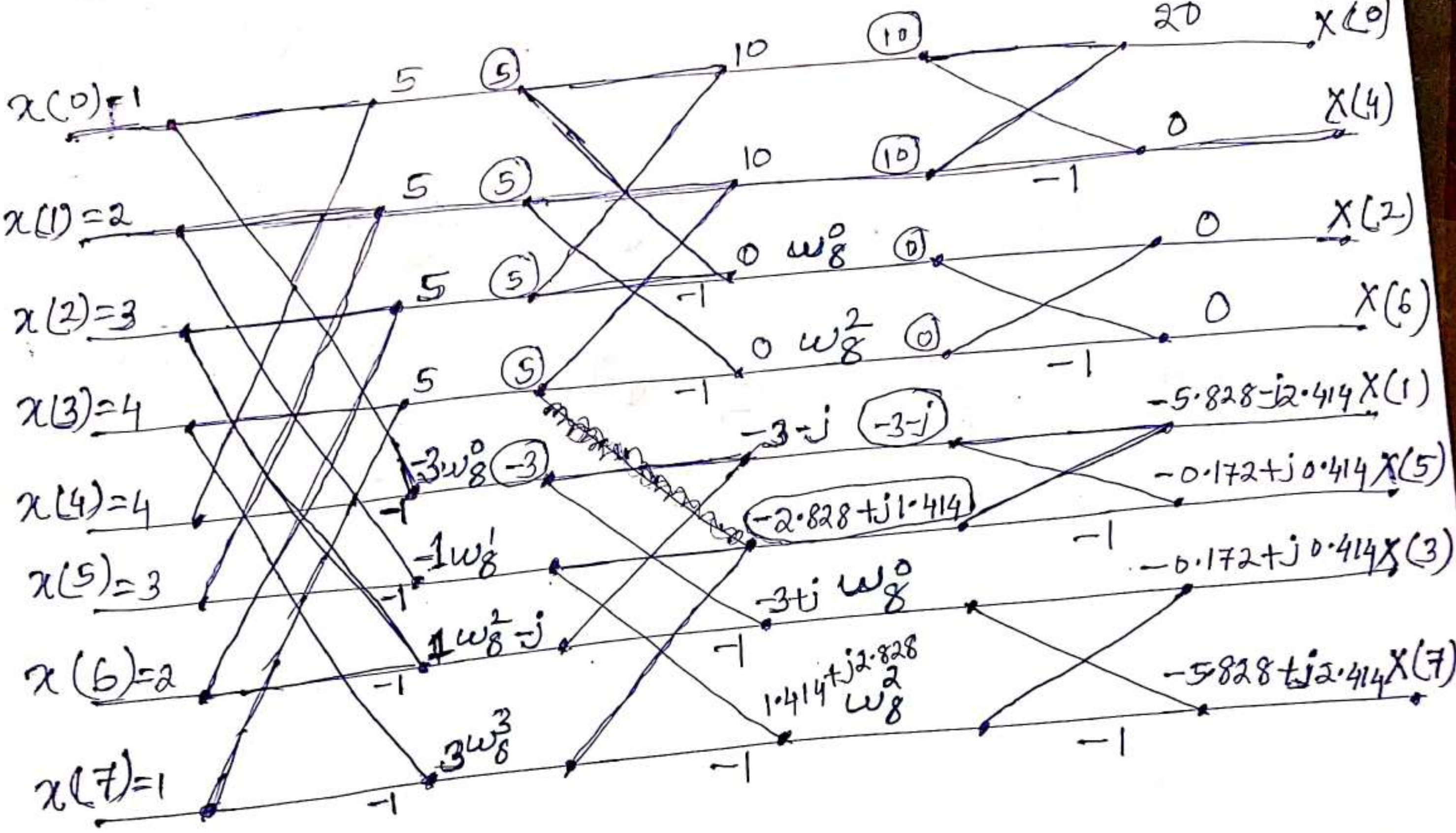
$1\times W_8^2=1\times -j=-j$, $j\times W_8^3=j\left(-\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)=\boxed{\frac{-j}{\sqrt{2}}+\frac{1}{\sqrt{2}}}$

So, Final DFT is $X(K)=\{7, -0.707-j0.707, -j, 0.707-j0.707, 1, 0.707+j0.707, j, -0.707+j0.707\}$

——x——x——

Que:- compute the DFT for the sequence {1, 2, 3, 4, 4, 3, 2, 1} using radix-2 DIF-FFT algorithm.

Sol:- Here $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$



Here $w_8^0 = 1$, $w_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$, $w_8^2 = -j$, $w_8^3 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$-3 \times w_8^0 = -3 \times 1 = -3$, $\quad -1 \times w_8^1 = -1\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) = \boxed{-\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}}$

$1 \times w_8^2 = 1 \times (-j) = -j$, $\quad 3 \times w_8^3 = 3 \times \left(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$

$= \boxed{\frac{-3}{\sqrt{2}} - j\frac{3}{\sqrt{2}}}$

$(-3+j) \times w_8^0 = (-3+j) \times 1 = \boxed{-3+j}$

$(1.414 + j2.828) \times w_8^2 = (1.414 + j2.828) \times (-j)$

$= \boxed{2.828 - j1.414}$

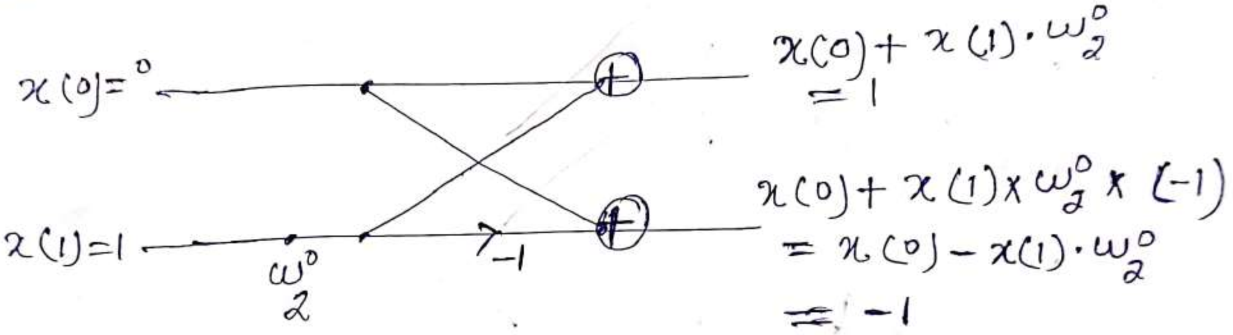
Scanned by CamScanner

$\therefore X(K) = \{20, -5.828 - j2.414, 0, -0.172 + j0.414,$
$0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

— x —

Que:- Find 2-point DIT-FFT Algorithm Butterfly Diagram. If $x(n) = \{0, 1\}$, Find $X(K)$

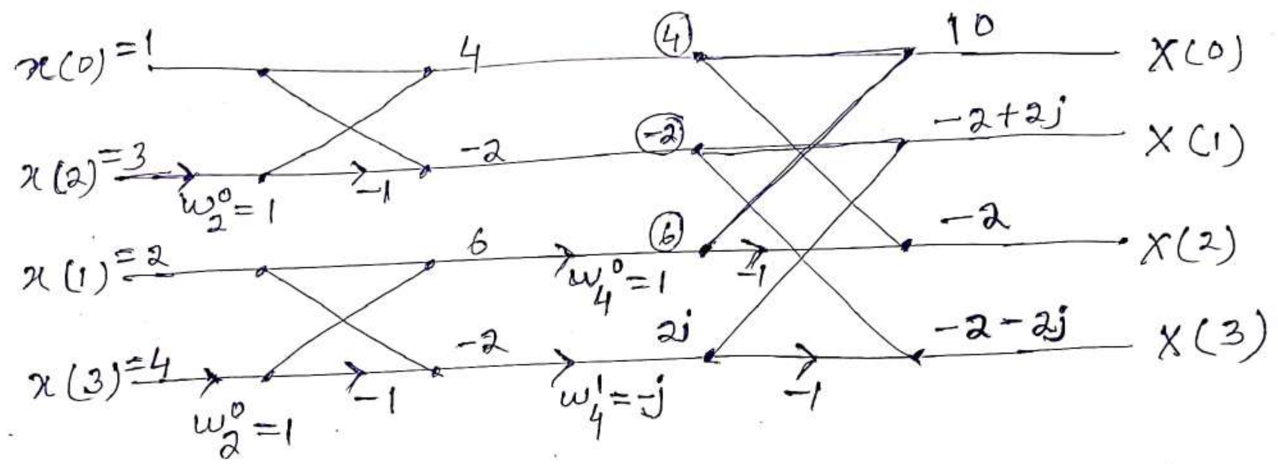
Soln:-



$W_2^0 = 1$

Que:- Find 4-point DIT FFT Butterfly Diagram where $x(n) = \{1, 2, 3, 4\}$

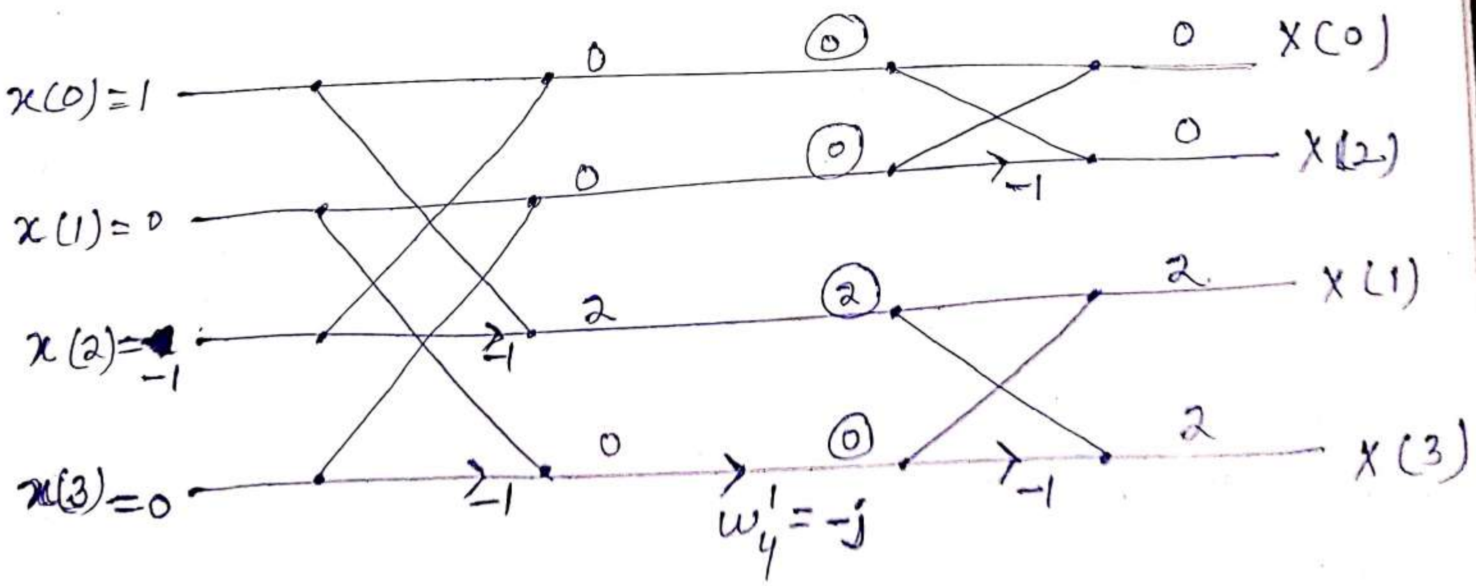
Soln:-



— x — x —

Que:- Find 4-point DIF-FFT Butterfly Diagram where $x(n) = \{1, 0, -1, 0\}$

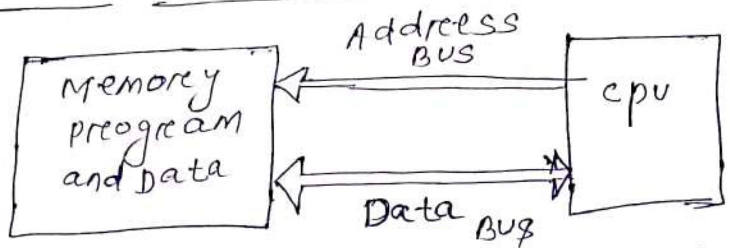
Soln:-



$\therefore X(K) = \{\underset{\uparrow}{0}, 2, 0, 2\}$

DIGITAL SIGNAL PROCESSING ARCHITECTURES :-

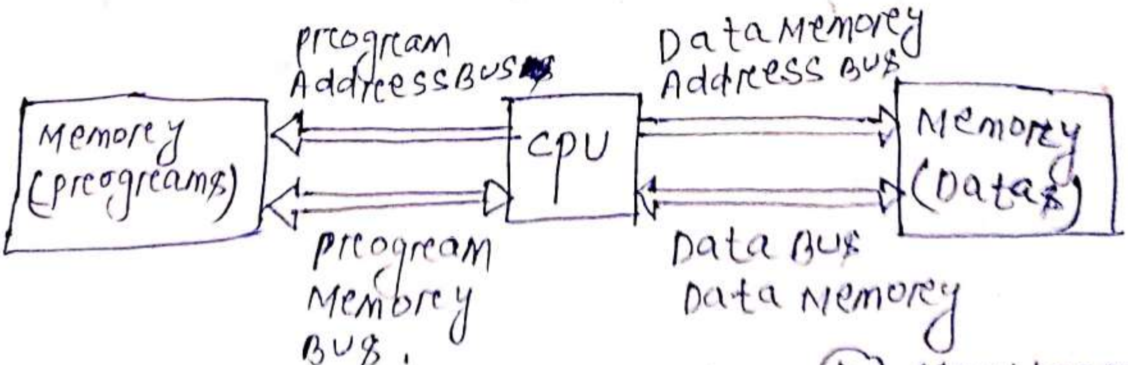
① Von-Neumann Architecture :-



↳ There is a common memory to store programs as well as data.
↳ The central processing unit can read an instruction or read/write data from/to the memory.
↳ Both can not occur at the same time as the instruction and data use the same bus system.
It has Data bus (Bidirectional), program/Address bus (unidirectional), control bus.

↳ The main characteristics of Von-Neuman Architecture is that it only posses one bus system. The same bus carries all information exchanged between cpu and peripherals including instruction codes as well as data processed cpu.

② Harvard Architecture :-



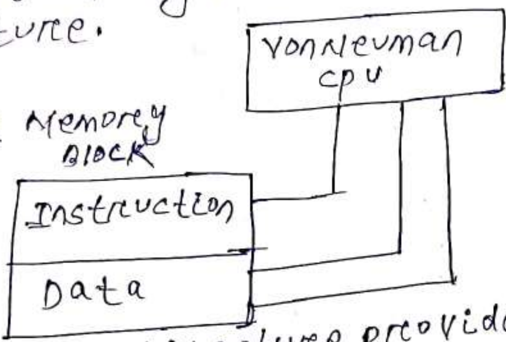
(i) Improve speed of processing. (ii) simultaneously operations can be performed.

(iii) Single memory pathway is there. (iv) physically separate pathway is also there.

——— x ———

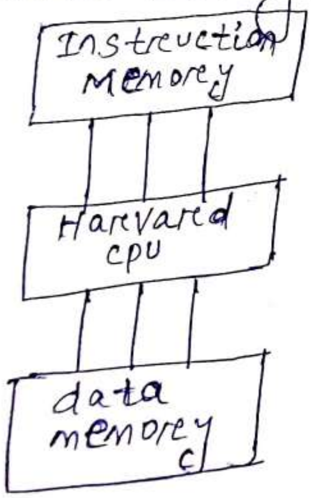
↳ In Harvard architecture, Data and code lie in Different memory Block. In Von Neuman Architecture, Data and code lie in same memory block.

↳ 1 data bus used for both instruction and data. CPU can perform only one operation at a time in Von Neuman Architecture.



All the buses i.e. Address bus, Data bus, control bus are accessing one memory block in VonNeuman Architecture.

↳ Harvard Architecture provides separate buses for both instruction and data. This architecture has data storage entirely contained within the cpu.



↳ In Von-Neuman Architecture, 2 set of clock cycles required. 1 cycle for Data fetch, and 1 cycle for instruction Fetch. whereas in Harvard Architecture single set of clock cycle is sufficient.

↳ In Von-Neuman Architecture, pipelining is not possible. In Harvard Architecture pipelining is possible.

↳ Von-Neuman Architecture is simple in design. whereas Harvard Architecture is complex in design.

——— x ———

5

FAST FOURIER TRANSFORM (FFT)

5.1 Introduction

The Fast Fourier Transform (FFT) does not represent a transform different from the DFT but they are special algorithms for speedier implementation of DFT. FFT requires a comparatively smaller number of arithmetic operations such as multiplications and additions than DFT. FFT also requires lesser computational time than DFT. The fundamental principle on which all these algorithms are based upon is that of decomposing the computation of the DFT of a sequence of length into successively smaller DFTs. The way in which this principle is implemented leads to a variety of different algorithms, all with comparable improvements in computational speed. Thus, we can say that DFT plays an important role in several applications of digital signal processing such as linear filtering, correlation analysis and spectrum analysis.

5.2 Fast Fourier Transform Algorithms

Direct computation of the DFT is less efficient because it does not exploit the properties of symmetry and periodicity of the phase factor $W_N = e^{-j2\pi/N}$

$W_N^K = e^{\frac{-j2\pi}{N} \cdot K}$

These properties are :

Symmetry property : $W_N^{K+N/2} = -W_N^K$

Periodicity property : $W_N^{K+N} = W_N^K$

As we already know that all computationally efficient algorithms for DFT are collectively known as FFT Algorithms and these algorithms exploit the above two properties of phase factor, W_N.

5.3 Classification of FFT Algorithms

A) According to the storage of the components of the intermediate vector, FFT algorithms are classified into two groups.

1. In-Place FFT algorithms
2. Natural Input-Output FFT algorithms.

1) In-Place FFT Algorithms. In this FFT algorithm, component of an intermediate vector can be stored at the same place as the corresponding component of the previous vector.

In-place FFT algorithms reduce the memory space requirement.

2) Natural Input-Output FFT Algorithms. In this FFT algorithm, both input and output are in natural order. It means both discrete-time sequence s(n) and its DFT, S(K) are in natural order. This type of algorithm consumes more memory spare for preservation of natural order of s(n) and S(K).

The disadvantage of an In-place FFT algorithm is that the output appears in an unnatural order necessitating proper shuffling of s(n) or S(K).

In-place FFT algorithms are superior to the Natural Input-output FFT algorithms although it needs shuffling of s(n) or S(K). This shuffling operation is known as Scrambling.

The scrambled value of an integer is defined as a new number generated by reversing the order of all bits in the equivalent binary number for that integer.

B) Another classification of FFT algorithms based on Decimation of s(n) or S(K). Decimation means decomposition into decimal parts.

On the basis of decimation process, FFT algorithms are of two types:

1. Decimation-in-Time FFT algorithms.
2. Decimation-in-Frequency FFT algorithms.

1) Decimation-in-Time (BIT) FFT Algorithms. In DIT FFT algorithms, the sequence s(n) will be broken up into odd numbered and even numbered subsequences.

2) Decimation-in-Frequency (DIF) FFT Algorithms. In DIF FFT algorithms, the sequence s(n) will be broken up into two equal halves.

Computation reduction factor of FFT algorithms

$$= \frac{\text{Number of computations required for direct DFT}}{\text{Number of computations required for FFT algorithm}}$$

$$= \frac{N^2}{\frac{N}{2}\log_2(N)}$$

5.4 Number of Stages in DFT Computation using FFT Algorithms

Number of stages in DFT computation using FFT algorithms depends upon the total number of points (N) in a given sequence.

For these algorithms, number of points in a discrete-time sequence,

$$N = 2^r \text{ where } r > 0.$$

r is the number of stages required for DFT computation via FFT algorithms.

Let us have a 8-point discrete-time sequence, $N = 8 = 2^3$. It requires three stages for DFT computations.

In Decimation-in-time (DIT) FFT algorithm, input discrete-time sequence s(n) is in Bit-reversed order but output, S(K) is in Natural order for in-place computation. In Decimation-in-frequency (DIF) FFT algorithm, input discrete-time sequence s(n) is in Natural order but its DFT is in Bit-reversed order for in-place computation. For in-place computation smaller memory space is required.

Generally, we use Radix-2 FFT algorithms. In Radix-2 FFT algorithms, original discrete-time sequence, s(n) is divided in two parts and DFT computation is done on each part separately and resultant of each parts added to get the overall discrete-frequency sequence.

In DIT FFT algorithm, original sequence s(n) is divided in even-numbered points and odd-numbered points. But in DIF FFT algorithm, original discrete-time sequence s(n) is divided in two parts as first half and second half. Fig. 5.1 illustrates the number of stages required in Appoint DFT computation via. DIT FFT algorithm (Here N = 8).

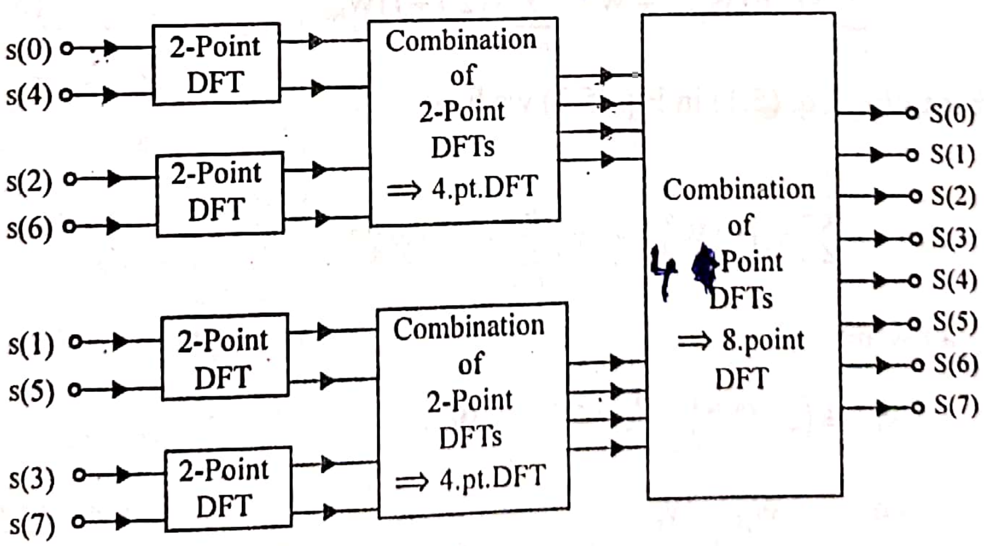


Fig. 5.1 *Three stages in N-point DFT computation via decimation-in-time FFT algorithm (N = 8)*

5.5 Decimation-in-time algorithm

This algorithm is also known as Radix-2 DIT FFT algorithm which means the number of output points N can be expressed as a power of 2, that is, $N = 2^M$, where M is an integer.

Let x(n) is an N-point sequence, where N is assumed to be a power of 2. Decimate or break this sequence into two sequences of length N/2, where one sequence consisting of the even-indexed values of x(n) and the other of odd-indexed values of x(n).

$$\text{i.e., } x_e(n) = x(2n) \qquad n = 0, 1 \ldots \frac{N}{2} - 1$$

$$x_o(n) = x(2n+1) \qquad n = 0, 1 \ldots \frac{N}{2} - 1 \qquad \text{....(5.1)}$$

The N-point DFT of x(n) can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad k = 0, 1, \ldots N-1 \qquad \text{....(5.2)}$$

Separating x(n) into even and odd indexed values of x(n), we obtain

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} + \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk} \qquad \text{....(5.3)}$$

Substituting Eq. (5.1) in Eq. (5.3) we have

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_N^{2nk} \qquad \text{...(5.4)}$$

we can write

$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/N/2} = W_{N/2}$$

$$\text{i.e.,} \qquad W_N^2 = W_{N/2} \qquad \text{...(5.5)}$$

Substituting Eq. (5.5) in Eq. (5.4) we get

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk}}_{\substack{N/2\text{-point DFT of even} \\ \text{indexed sequence}}} + \underbrace{W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n)\, W_{N/2}^{nk}}_{\substack{N/2\text{-point DFT of odd} \\ \text{indexed sequence}}} \qquad \text{....(5.6)}$$

$$= X_e(k) + W_N^k X_o(k) \qquad \text{....(5.7)}$$

Each of the sums in Eq. (5.6) is an $\frac{N}{2}$ point DFT, the first sum being the $\frac{N}{2}$-Point DFT of the even -indexed sequence and the second being the $\frac{N}{2}$-point DFT of the odd-indexed sequence. Although the index k ranges from k=0, 1....N – 1, each of the sums are computed only for k =0, 1.... $\frac{N}{2}-1$, since $X_e(k)$ and $X_0(k)$ are periodic in k with period $\frac{N}{2}$. After the two DFTs are computed, they are combined according to Eq. (5.7) to get the N-point DFT of X(k). So the Eq. (5.7) holds good for the values of k = 0, 1... $\frac{N}{2}-1$.

For $k \geq N/2$

$$W_N^{k+N/2} = -W_N^k \qquad \text{...(5.8)}$$

Now X(k) for $k \geq N/2$ is given by

$$X(k) = X_e\left(k - \frac{N}{2}\right) - W_N^{k-N/2} X_0\left(k - \frac{N}{2}\right) \text{for } k = \frac{N}{2}, \frac{N}{2}+1, \ldots N-1 \qquad \text{...(5.9)}$$

Let us take N = 8. Then $X_e(k)$ and $X_0(k)$ are 4-point ($\because$ N/2) DFTs of even-indexed sequence $x_e(n)$ and odd-indexed sequence $x_0(n)$ respectively,

where

$$\begin{array}{ll} x_e(0) = x(0) & x_0(0) = x(1) \\ x_e(1) = x(2) & x_0(1) = x(3) \\ x_e(2) = x(4) & x_0(2) = x(5) \\ x_e(3) = x(6) & x_0(3) = x(7) \end{array}$$

From Eq.(5.7) and Eq. (5.9) we have

$$\boxed{\begin{aligned} X(k) &= X_e(k) + W_8^k X_0(k) \quad \text{for } 0 \leq k \leq 3 \\ &= X_e(k-4) - W_8^{k-4} X_0(k-4) \quad \text{for } 4 \leq k \leq 7 \end{aligned}} \qquad \text{...(5.10)}$$

By substituting different values of k we get

$$\begin{array}{ll} X(0) = X_e(0) + W_8^0 X_0(0); & X(4) = X_e(0) - W_8^0 X_0(0) \\ X(1) = X_e(1) + W_8^1 X_0(1); & X(5) = X_e(1) - W_8^1 X_0(1) \\ X(2) = X_e(2) + W_8^2 X_0(2); & X(6) = X_e(2) - W_8^2 X_0(2) \\ X(3) = X_e(3) + W_8^3 X_0(3); & X(7) = X_e(3) - W_8^3 X_0(3) \end{array} \qquad \text{....(5.11)}$$

From the above set of equations we can find that X(0) & X(4), X(1) & X(5), X(2) & X(6), X(3) & X(7) have same inputs. X(0) is obtained by multiplying $X_o(0)$ with W_8^0 and adding the product to $X_e(0)$. Similarly X(4) is obtained by multiplying $X_o(0)$ with W_8^0 and subtracting the product from $X_e(0)$.

This operation can be represented by a butterfly diagram as shown in Fig. 5.2

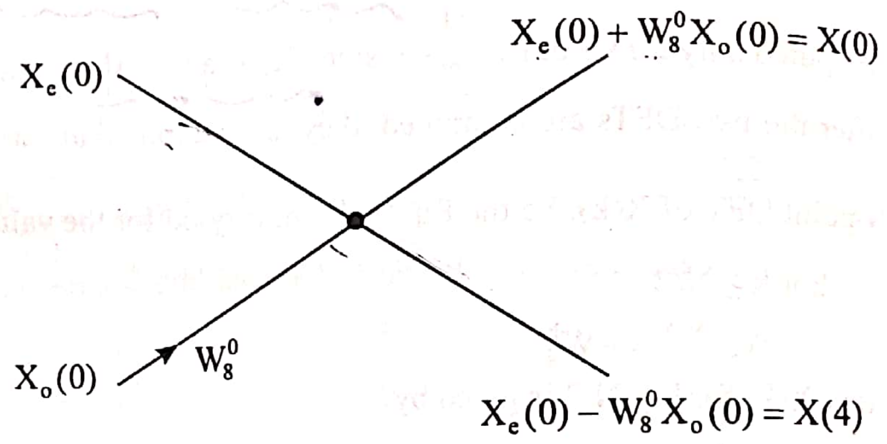


Fig. 5.2 *Flow graph of butterfly diagram for Eq. 5.11*

Now the values X(k) for k = 1, 2, 3, 5, 6, 7 can be obtained and an 8-point DFT flowgraph can be constructed from two 4-point DFTs as shown in Fig. 5.3

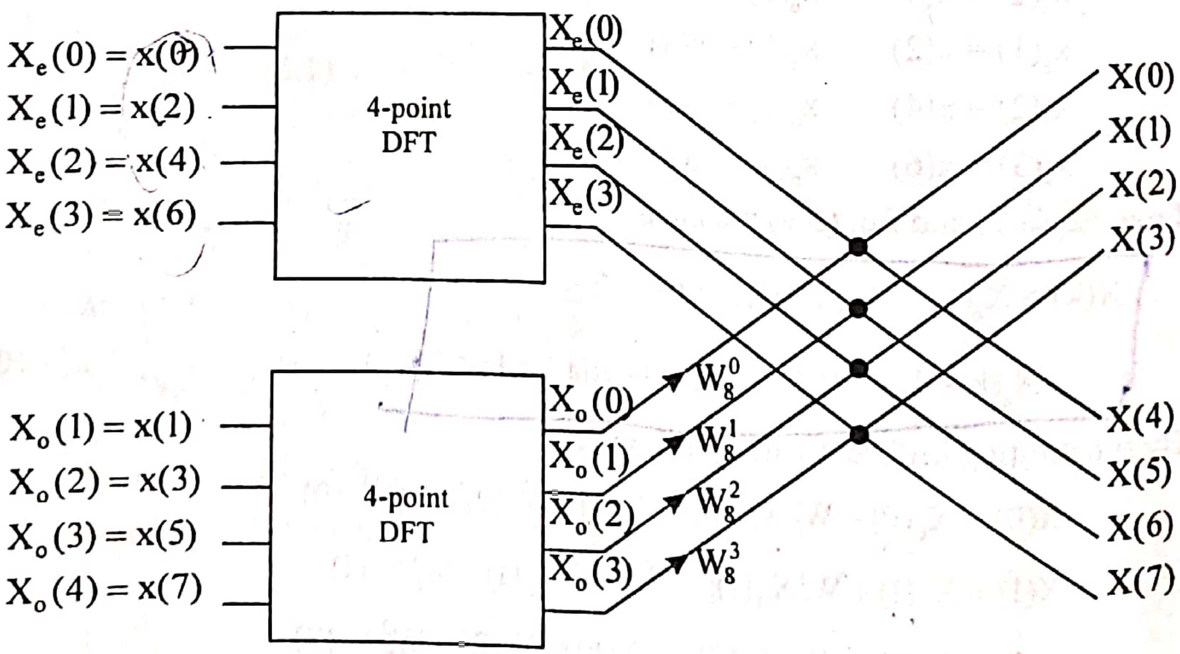


Fig. 5.3 *Construction of an 8-point DFT from two 4 point DFTs*

From the Fig. 5.3 we can find that initially the sequence x(n) is shuffled into even-indexed sequence $x_e(n)$ and odd-indexed sequence $x_o(n)$ and then transformed to give $X_e(k)$ and $X_o(k)$. For $k = 0, 1,2, 3$ the values $X_e(k)$ and $X_o(k)$ are combined according to Eqs. (5.11) and using butterfly structure shown in Fig.5.2 the 8-point DFT is obtained. The inputs to the butterfly is separated by $\frac{N}{2}$ samples i.e., 4 samples and the powers of the twiddle factors associated in this set of butterflies are in natural order.

Now we apply the same approach to decompose each of $\frac{N}{2}$ sample DFT. This can be done by dividing the sequence $x_e(n)$ and $x_o(n)$ into two sequences consisting of even and odd members of the sequences. The $\frac{N}{2}$ point DFTs can be expressed as a combination of $\frac{N}{4}$-point DFTs.

i.e. $X_e(k)$ for $0 \le k \le \frac{N}{2} - 1$ can be written as

$$X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k) \text{ for } 0 \le k \le \frac{N}{4} - 1$$

$$= X_{ee}\left(k - \frac{N}{4}\right) - W_N^{2(k-N/4)} X_{eo}\left(k - \frac{N}{4}\right) \text{for } \frac{N}{4} \le k \le \frac{N}{2} - 1 \quad \text{....(5.12)}$$

where $X_{ee}(k)$ is the $\frac{N}{4}$-point DFT of the even members of $x_e(n)$ and $X_{eo}(k)$ is the $\frac{N}{4}$-point of DFT of the odd members of $x_e(n)$.

In the same way

$$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k) \text{ for } 0 \le k \le \frac{N}{4} - 1$$

$$= X_{oe}\left(k - \frac{N}{4}\right) - W_N^{2(k-N/4)} X_{oo}\left(k - \frac{N}{4}\right) \text{for } \frac{N}{4} \le k \le \frac{N}{2} - 1 \quad \text{...(5.13)}$$

where $X_{oe}(k)$ is the $\frac{N}{4}$-point DFT of the even members of $x_o(n)$ and $X_{oo}(k)$ is the $\frac{N}{4}$-point DFT of the odd members of $x_o(n)$.

For N = 8

the sequence $x_e(n)$ can be divided into even and odd indexed sequences as

$x_{ee}(0) = x_e(0); x_{ee}(1) = x_e(2)$

$x_{eo}(0) = x_e(1); x_{eo}(1) = x_e(3)$

Now from Eq. (5.12) we have

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$

$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$

$$X_e(2) = X_{ee}(0) + W_8^0 X_{eo}(0)$$

$$X_e(3) = X_{ee}(1) + W_8^2 X_{eo}(1) \qquad ...(5.14)$$

where $X_{ee}(k)$ is the 2 point DFT of even members of $x_e(n)$ and $X_{eo}(k)$ is the 2-point DFT of odd members of $x_e(n)$.

Similarly

the sequence $x_o(n)$ can be divided into even and odd membered sequences as

$x_{oe}(0) = x_o(0)\ x_{oe}(1) = x_o(2)$

$x_{oo}(0) = x_o(1)\ x_{oo}(1) = x_o(3)$

From the Eq. (5.13) we can obtain

$$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$

$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$

$$X_o(2) = X_{oe}(0) + W_8^0 X_{oo}(0)$$

$$X_o(3) = X_{oe}(1) + W_8^2 X_{oo}(1) \qquad ...(5.15)$$

where

$X_{oe}(k)$ is the 2-point DFT of the even members of $x_o(n)$,

$X_{oo}(k)$ is the 2-point DFT of the odd members of $x_o(n)$.

Fig. 5.4 shows the resulting flow graph when the four-point DFTs of Fig. 5.3 are evaluated as in Eq. (5.14) and Eq. (5.15)

From the Fig. 5.4 we find that the input sequence is again reordered, the input samples to each butterfly are separated by $\frac{N}{4}$ samples i.e., 2 samples and there are two sets of butterflies. In each set of butterflies the twiddle factor exponents are same and separated by two.

For the more general case, we could proceed by decomposing the $\frac{N}{4}$ - point transforms in Eq. (5.12) and Eq. (5.13) into $\frac{N}{8}$ -point transforms and continue until you left with only 2-point transforms. Each decomposition is called a stage, and the total number of stages is given by $M = \log_2 N$. The 8-point DFT requires 3 stages. So far we have seen the decomposition for stage 3 and stage 2. For stage 1 the two point DFT can be easily found by adding and subtracting the input sequences as the twiddle factor associated with first stage is $W_8^0 = 1$, i.e.,

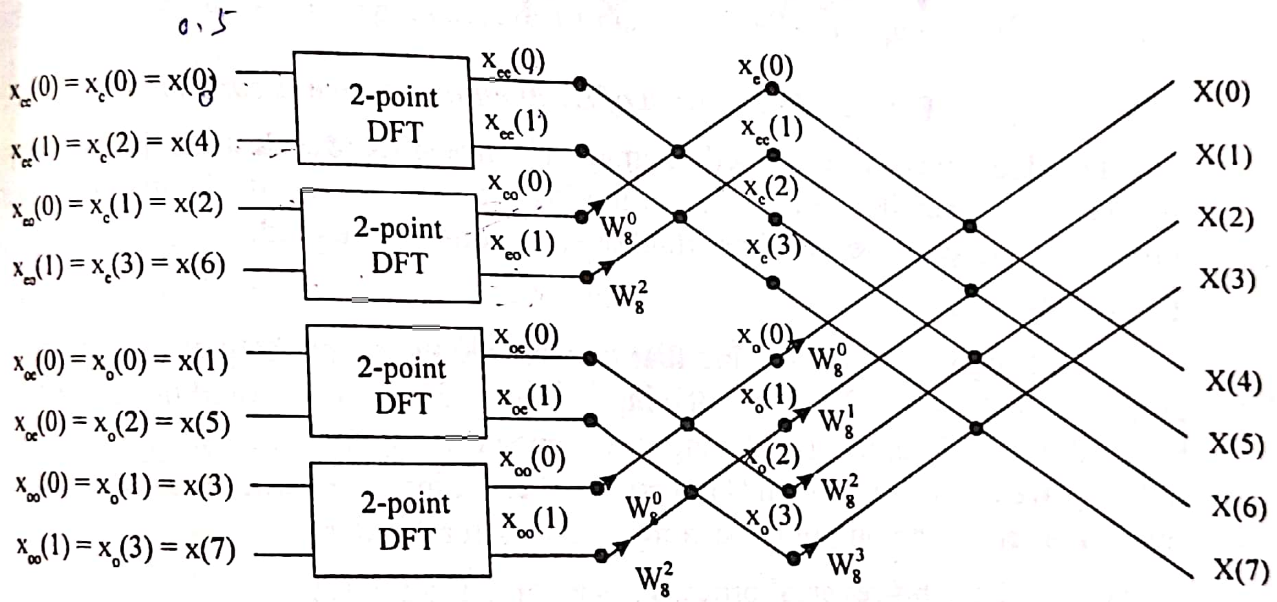


Fig. 5.4 *Construction of 8 point DFT from two 4 point DFTs and 4 point DFT from two point DFTs.*

the first stage involves no multiplication but addition and subtracting. Now we have

$$X_{ee}(0) = x_{ee}(0) + x_{ee}(1) = x_e(0) + x_e(2) = x(0) + x(4)$$

$$X_{ee}(1) = x_{ee}(0) - x_{ee}(1) = x_e(0) + x_e(2) = x(0) - x(4) \qquad ....(5.16)$$

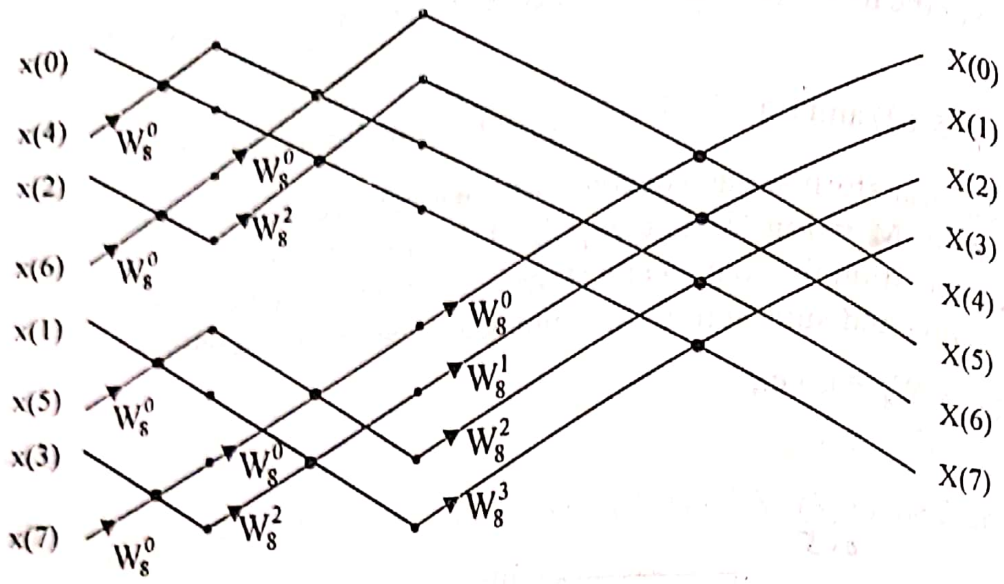


Fig. 5.5. *Flow graph of Decimation-in-time algorithm.*

The algorithm has been called decimation in time since at each stage, the input sequence is divided into smaller sequences i.e. the input sequences are decimated at each stage. From the flow graph several important observations can be made.

1. Bit Reversal

In DIT algorithm we can find that in order for the output sequence to be in natural order (i.e., X(k), k = 0, 1... N – 1) the input sequence had to be stored in a shuffled order. For an 8-point DIT algorithm the input sequence is x(0), x(4), x(2), x(6), x(1), x(5), x(3) and x(7). We can see that when N is a power of 2, the input sequence must be stored in bit-reversal order for the output to be computed in natural order.

For N = 8 the bit-reversal process is shown in table 5.1.

Table 5.1 Bit-reversal process for N = 8

Input sample index	Binary representation	Bit reversed binary	Bit reversed sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

5.5 Steps of *radix* - 2 DIT-FFT algorithm

1. The number of input samples $N = 2^M$, where, M is an integer.
2. The input sequence is shuffled through bit-reversal.
3. The number of stages in the flowgraph is given by $M = \log_2 N$.
4. Each stage consists of $\frac{N}{2}$ butterflies.
5. Inputs/outputs for each butterfly are separated by 2^{m-1} samples, where m represents the stage index, i.e., for first stage m = 1 and for second stage m = 2 so on.
6. The number of complex multiplications is given by $\frac{N}{2}\log_2 N$.
7. The number of complex additions is given by $N \log_2 N$.
8. The twiddle factor exponents are a function of the stage index m and is given by

$$k = \frac{Nt}{2^m} \qquad t = 0, 1, 2, \ldots\ 2^{m-1} - 1. \qquad \ldots(5.17)$$

9. The number of sets or sections of butterflies in each stage is given by the formula 2^{M-m}.
10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with m is repeated is given by 2^{M-m}.

Table 5.2 Phase Rotation Factors for Quick Computation

Number of points in DFT, N	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
4 No. of stages=2	Twiddle factor not required	W_4^0, W_4^1	–	–	–
8 No. of stages = 3	Twiddle factor not required	W_8^0, W_8^2	W_8^0, W_8^1 W_8^2, W_8^3		
16 No. of stages = 4	Twiddle factor not required	W_{16}^0, W_{16}^4	W_{16}^0, W_{16}^2 W_{16}^4, W_{16}^6 [illegible]	W_{16}^0, W_{16}^1 W_{16}^2, W_{16}^3 W_{16}^6, W_{16}^7	
32 No. of stages = 5	Twiddle factor not required	W_{32}^0, W_{32}^8	W_{32}^0, W_{32}^4 W_{32}^8, W_{32}^{12}	W_{32}^0, W_{32}^2 W_{32}^{14}	W_{32}^0, $W_{32}^1, \ldots$ W_{32}^{15}

Example 5.1

Draw the Flow graph of 16-point DIT-FFT.

Solution

1. The number of input Samples, $N = 16$
2. The input sequence is shuffled through bit-reversal shown in table 5.3 and applied as input to the flow graph.

Table 5.3 Bit-reversal process

Index	Binary Representation	Bit-reversal Order	Bit-reversal Index
0	0000	0000	0
1	0001	1000	8
2	0010	0100	4
3	0011	1100	12
4	0100	0010	2
5	0101	1010	10
6	0110	0110	6
7	0111	1110	14
8	1000	0001	1
9	1001	1001	9
10	1010	0101	5
11	1011	1101	13
12	1100	0011	3
13	1101	1011	11
14	1110	0111	7
15	1111	1111	15

3. The number of stages $M = \log_2 16 = 4$.
4. The number of butterflies per stage is $\frac{N}{2} = 8$.
5. The inputs/outputs for each butterfly in stage m is separated by 2^{m-1} samples.

Stage 1 Inputs/outputs for each butterfly are separated by 1 sample.

Stage 2 Inputs/outputs for each butterfly are separated by 2 samples.

Stage 3 Inputs/outputs for each butterfly are separated by 4 samples.

Stage 4 Inputs/outputs for each butterfly are separated by 8 samples.

6. The number of complex multiplications is given by

$$\frac{N}{2}\log_2 N = 8\log_2 16 = 32$$

7. The number of complex additions is given by $16\log_2 16 = 64$
8. The number of sets or sections of butterflies in each stage is given by 2^{M-m}
For Stage 1 the number of sets of butterflies are $2^{4-1} = 8$
For Stage 2 the number of sets of butterflies are $2^{4-2} = 4$
For Stage 3 the number of sets of butterflies are $2^{4-3} = 2$
For Stage 4 there is only one set of butterflies.
9. The twiddle factor exponents for each stage are given by

$$\boxed{k = \frac{Nt}{2^m} \quad t = 0, 1, 2, \ldots.\ 2^{m-1} - 1.}$$

For Stage 1 the exponent is 0 $\rightarrow K = \frac{16 \cdot 0}{1} = 0,$
For Stage 2 the exponents are 0, 4 $\rightarrow K = \frac{16 \cdot 0}{2} = 0, \quad K = \frac{16 \cdot 1}{4} = 4.$
For Stage 3 the exponents are 0, 2, 4, 6
For Stage 4 the exponents are 0, 1, 2, 3, 4, 5, 6, 7

10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with m is repeat is given by 2^{M-m}.

For stage 1 ERF = 8

For stage 2 ERF = 4

For stage 3 ERF = 2

For stage 4 ERF = 1.

From the steps 8, 9, 10 we can draw the following conclusion.

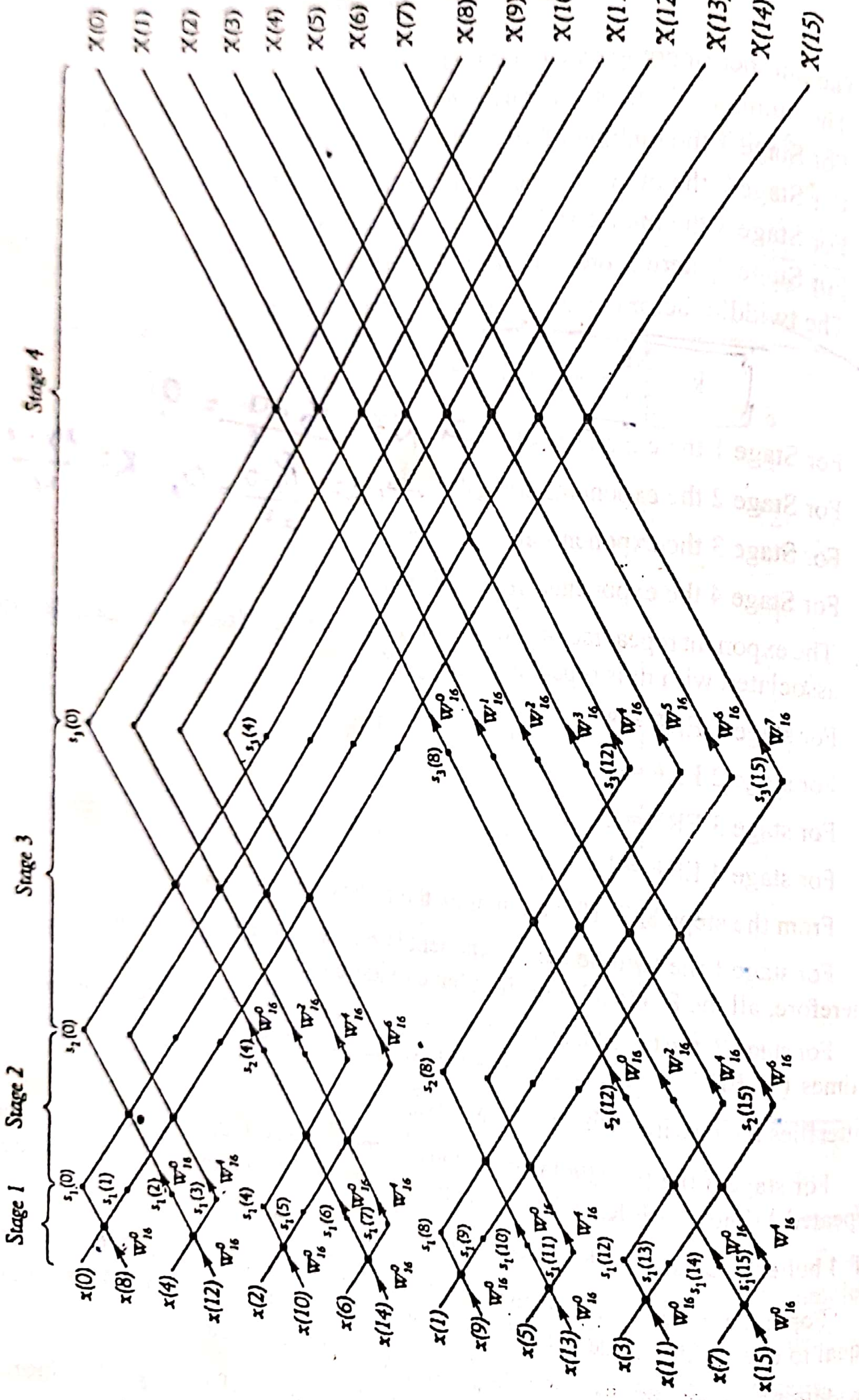
For stage 1 the twiddle factor exponent is zero and is repeated 8 times. ($\because$ ERF = 8). Therefore, all the 8 sets of butterflies have twiddle facts

For stage 2, the twiddle factor exponents sequence is 0, 4 and this sequence is repeated 4 times ($\because$ ERF = 4), i.e., all the 4 sets of butterflies where each set consists of two butterflies have twiddle factors as W_{16}^0, W_{16}^4.

For stage 3 the twiddle factor exponents sequence is 0, 2, 4, 6 and this sequence is repeated 2 times ($\because$ E.R.F = 2), i.e., all the two sets of butterflies where each set consists of 4 butterflies have twiddle factors as $W_{16}^0, W_{16}^2, W_{16}^4, W_{16}^6$.

For stage 4 the twiddle factor exponents sequence is 0, 1, 2, 3, 4, 5, 6, 7 and ERF is equal to one. In this stage the only set of butterflies which consists of 8 butterflies have twiddle factor as $W_{16}^0, W_{16}^1, W_{16}^2, W_{16}^3, W_{16}^4, W_{16}^5, W_{16}^6, W_{16}^7$

Using the above steps the complete flowgraph of 16 point DFT using D1T algorithm is drawn as shown in Fig. 5.6.



Fog 5.6 : *Flow Graph of 16-point DIT-FFT algorithm*

Example 5.2

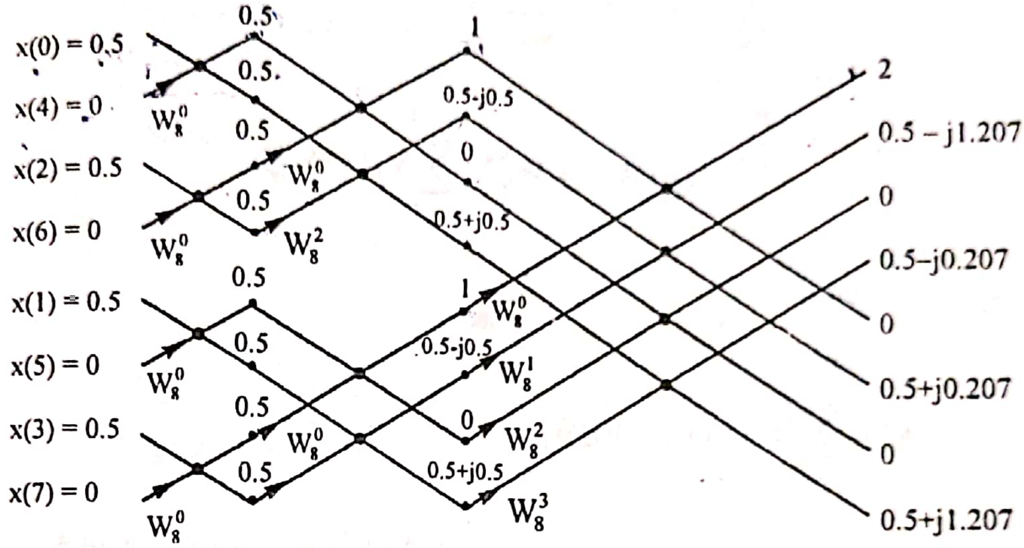
Compute the eight-point DFT of the sequences x(n) = {0.5, 0.5, 0.5, 0, 0, 0} using the radix-2 DIT algorithm.

x(n)

Solution

The twiddle factors are

$W^0 = 1$; $W^1 = 0.707 - j0.707$; $W^2 = -j$; $W^3 = -0.707 - j0.707$



X(k) = {2, 0.5 – j 1.207, 0, 0.5 – j 0.207, 0, 05. + j 0.207, 0, 0.5 + j 1.207 }

5.6 Decimation-in-frequency algorithm

DIT algorithm is based on the decomposition of the DFT computation by forming smaller and smaller subsequences of the sequence x(n). In DIP algorithm the output sequence X(k) is divided into smaller and smaller subsequences. In this algorithm the input sequence x(n) is partitioned into two sequences each of length $\frac{N}{2}$ samples. The first sequence $x_1(n)$ consists of first $\frac{N}{2}$ samples of x(n) and the second sequence $x_2(n)$ consists of the last $\frac{N}{2}$ samples of x(n) i.e.,

$$x_1(n) = x(n),\ n = 0, 1, 2, \ldots N/2 - 1 \qquad \ldots(5.18)$$

$$x_2(n) = x(n + N/2)\ n = 0, 1, 2, \ldots N/2 - 1 \qquad \ldots(5.19)$$

i.e., If N = 8 the first sequence $x_1(n)$ has values for $0 \le n \le 3$ and $x_2(n)$ has values for $4 \le n \le 7$.

The N-point DFT of x(n) can be written as

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\,W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)\,W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n)\,W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)\,W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x_2(n)\,W_N^{nk}$$

when k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)]W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)]W_{N/2}^{nk} \qquad \text{...(5.20)}$$

$(\because\ W_N^2 = W_{N/2})$

Eq. (5.20) is the $\frac{N}{2}$-point DFT of the $\frac{N}{2}$-point sequence obtained by adding first half and the last half of the input sequence.

when k is odd $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)]W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)]W_N^{2kn}W_{N/2}^{nk} \qquad \text{....(5.21)}$$

Eq. (5.21) is the $\frac{N}{2}$-point of DFT of the sequence obtained by subtracting the second half of the input sequence from the first half and multiplying the resulting sequence by W_N^n

Eq. (5.20) and Eq. (5.21) show that the even and odd samples of the DFT N can be obtained from the $\frac{N}{2}$-point DFTs of $f(n)$ and g(n) respectively

where $f(n) = x_1(n) + x_2(n) \qquad n = 0, 1 \ldots \frac{N}{2} - 1$

$$g(n) = [x_1(n) - x_2(n)]\ W_N^n \qquad n = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots (5.22)$$

The Eq. (5.22) can be represented by a butterfly as shown in Fig. 5.7. This is the basic operation of DIF algorithm.

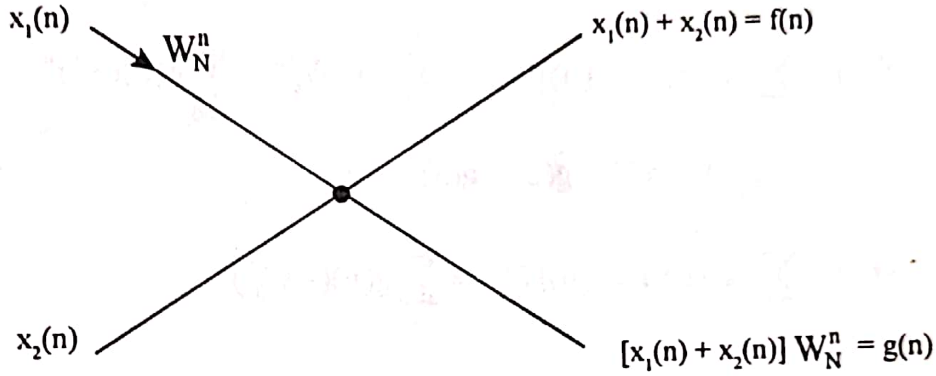


Fig 5.7 *Flow graph of basic butterfly diagram for DIF algorithm*

From Eq. (5.20), for N = 8, we have

$$X(0) = \sum_{n=0}^{3}[x_1(n) + x_2(n)] = \sum_{n=0}^{3} f(n) = f(0) + f(1) + f(2) + f(3) \qquad \ldots (5.23)$$

$$X(2) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{2n} = \sum_{n=0}^{3} f(n)W_8^{2n}$$

$$= f(0) + f(1)W_8^2 - f(2) - f(3)W_8^2$$

$$\boxed{\begin{array}{l} W_8^4 = (e^{-j2\pi/8})^4 = e^{j\pi} = -1 \\ W_8^8 = (e^{-j2\pi/8})^8 = e^{j2\pi} = 1 \end{array}}$$

$$X(4) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{4n} = \sum_{n=0}^{3} f(n)W_8^{4n} = \sum_{n=0}^{3} f(n)(-1)^n$$

$$= f(0) - f(1) + f(2) - f(3) \qquad \ldots (5.24)$$

$$X(6) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{6n} = \sum_{n=0}^{3} f(n)(-W_8^2)^n$$

$$= f(0) - f(1)\ W_8^2 - f(2) + f(3)\ W_8^2 \quad ....(5.25)$$

From Eq. (5.21) we have

$$X(1) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^n = \sum_{n=0}^{3} g(n) = g(0) + g(1) + g(2) + g(3) \quad ....(5.26)$$

$$X(3) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{3n} = \sum_{n=0}^{3} g(n)W_8^{2n}$$

$$= g(0) + g(1)\ W_8^2 - g(2) - g(3)\ W_8^2 \quad ....(5.27)$$

$$X(5) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{5n} = \sum_{n=0}^{3} g(n)W_8^{4n} = \sum_{n=0}^{3} g(n)(-1)^n$$

$$= g(0) - g(1) + g(2) - g(3) \quad ...(5.28)$$

$$X(7) = \sum_{n=0}^{3}[x_1(n) + x_2(n)]W_8^{7n} = \sum_{n=0}^{3} g(n)(-W_8^2)^n$$

$$= g(0) - g(1)\ W_8^2 - g(2) + g(3)\ W_8^2 \quad ...(5.29)$$

We have seen that the even-valued samples of X(k) can be obtained from the 4-point DFT of the sequence f(n) where.

$$f(n) = x_1(n) + x_2(n) \qquad n = 0, 1 \ldots \frac{N}{2} - 1$$

i.e.,

$$f(0) = x_1(0) + x_2(0)$$
$$f(1) = x_1(1) + x_2(1)$$
$$f(2) = x_1(2) + x_2(2)$$
$$f(3) = x_1(3) + x_2(3) \quad ....(5.30)$$

The odd-valued samples of X(k) can be obtained from the 4-point DFT of the sequence g(n) where $g(n) = [x_1(n) - x_2(n)]\ W_8^n$

i.e.,

$$g(0) = [x_2(0) - x_2(0)]\ W_8^0$$

$$g(1) = [x_1(1) - x_2(1)]\ W_8^1$$

$$g(2) = [x_1(2) - x_2(2)]\ W_8^2$$

$$g(3) = [x_1(3) - x_2(3)]\ W_8^3 \quad ....(5.31)$$

Using the above information and the butterfly structure shown in Fig. 5.7 we can draw the flow graph of 8-point DFT shown in Fig. 5.8.

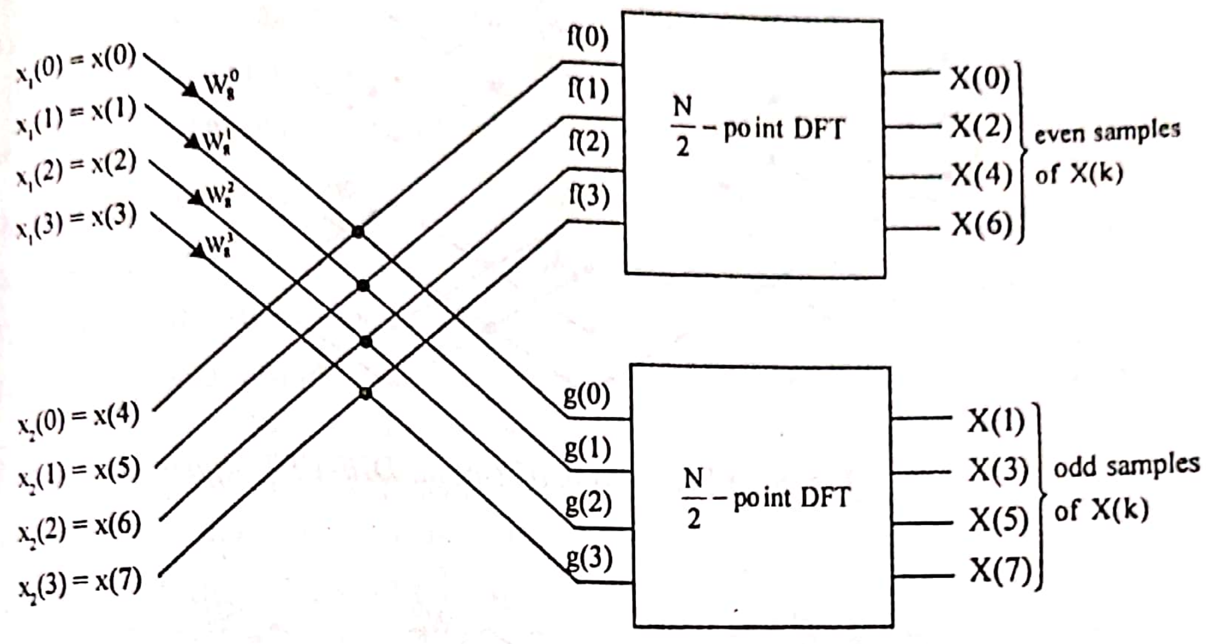


Fig. 5.8 *Reduction of an 8 point DFT to two 4 point DFTs by decimation in frequency*

Now each $\frac{N}{2}$-point DFT can be computed by combining the first half and the last half of the input points for each of the $\frac{N}{2}$-point DFTs and then computing $\frac{N}{4}$-point DFTs. For the 8-point DFT example the resultant flow graph is shown in Fig.5.9

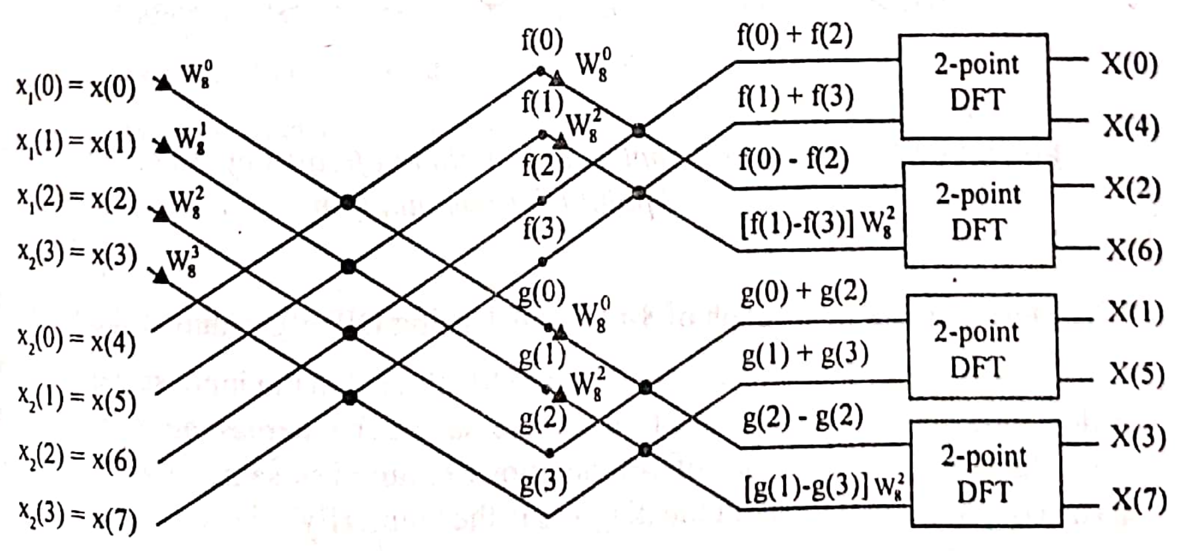


Fig. 5.9 *Flow graph of decimation in frequency decomposition of an 8-point DFT into four 2-point DFT computations*

The 2-point DFT can be found by adding & subtracting the input points. The Fig.5.9 can be further reduced as in Fig. 5.10.

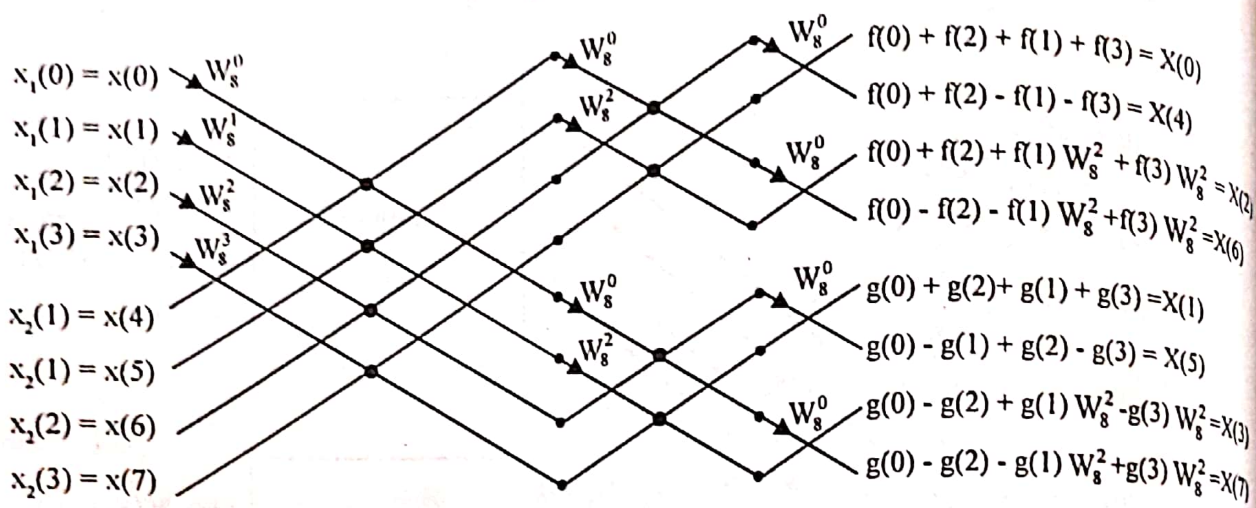


Fig. 5.10 *Flow graph of 8-point DIF-FFT algorithm*

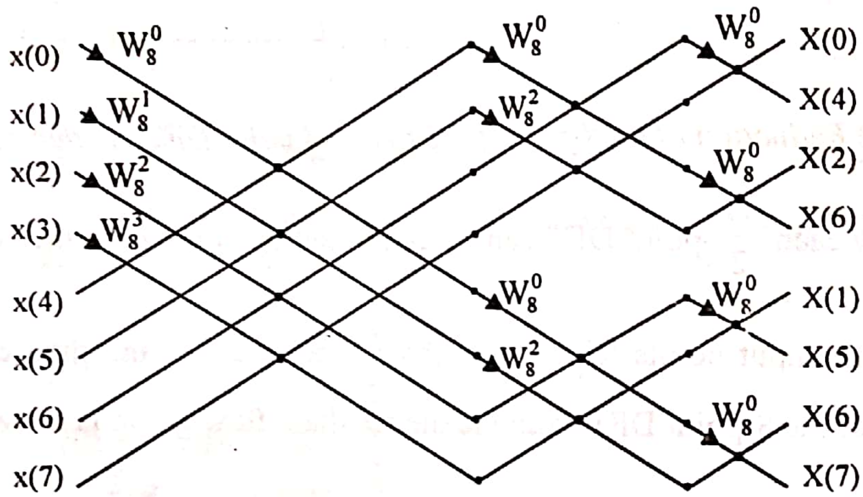


Fig. 5.11 *Flow graph of Complete decimation in frequency decomposition of an 8 point DFT computation*

The complete flow graph of 8-point DFT using DIF algorithm is shown in Fig. 5.11. From the Fig. 5.11 we observe that for DIF algorithm the input sequence is in natural order, while the output sequence is in bit reversal order, whereas the reverse is true for the DIT algorithm. The number of computations required is same as DIT algorithm. The basic computational block in the diagram is the "butterfly" shown in Fig. 5.12.

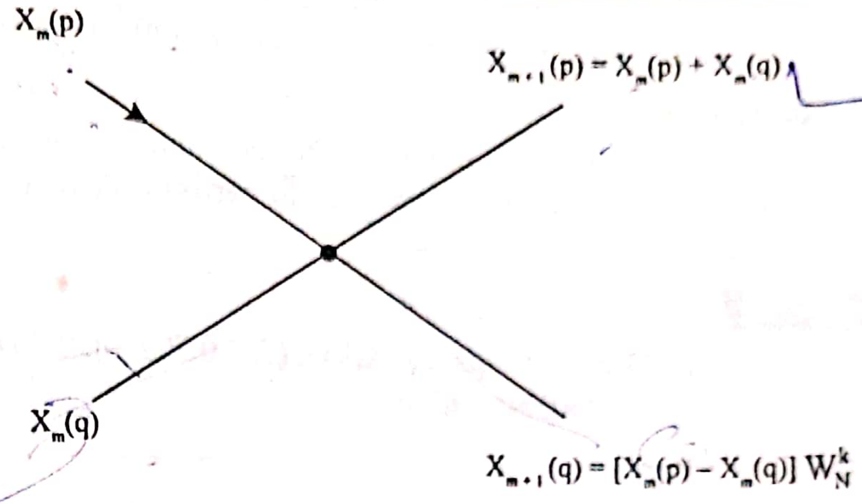


Fig. 5.12 *Basic Computational diagram for DIF-FFT*

Like DIT algorithm, DIF algorithm also in-place algorithm where the same locations are use to store both the input and output sequences.

5.7 Steps for Radix - 2 DIF-FFT algorithm

1. The number of input samples N = 2 , where, M is number of stages.
2. The input sequence is in natural order.
3. The number of stages in the flow graph is given by $M = \log_2 N$.
4. Each stage consists of $\frac{N}{2}$ butterflies.
5. Inputs/outputs for each butterfly are separated by 2^{M-m} samples, where m represents the stage index i.e., for first stage m = 1 and for second stage m = 2 so on.
6. The number of complex multiplications is given by
7. The number of complex additions is given by $N \log_2 N$.
8. The twiddle -factor exponents are a function of the stage index m and is given by

$$k = \frac{Nt}{2^{M-m+1}} \quad t = 0, 1, 2, \ldots 2^{M-m} \qquad \ldots (5.32)$$

9. The number of sets or sections of butterflies in each stage is given by the formula 2^{m-1}.
10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with m is repeated is given by 2

5.8 Differences and similarities between DIT and DIF algorithms

Differences

1. For decimation-in-time (DIT), the input is bit-reversed while the output is in natural order. Whereas, for decimation-in-frequency the input is in natural order while the output is bit reversed order.

2. The DIF butterfly is slightly different from the DIT wherein DIF the complex multiplication takes place after the add-subtract operation.

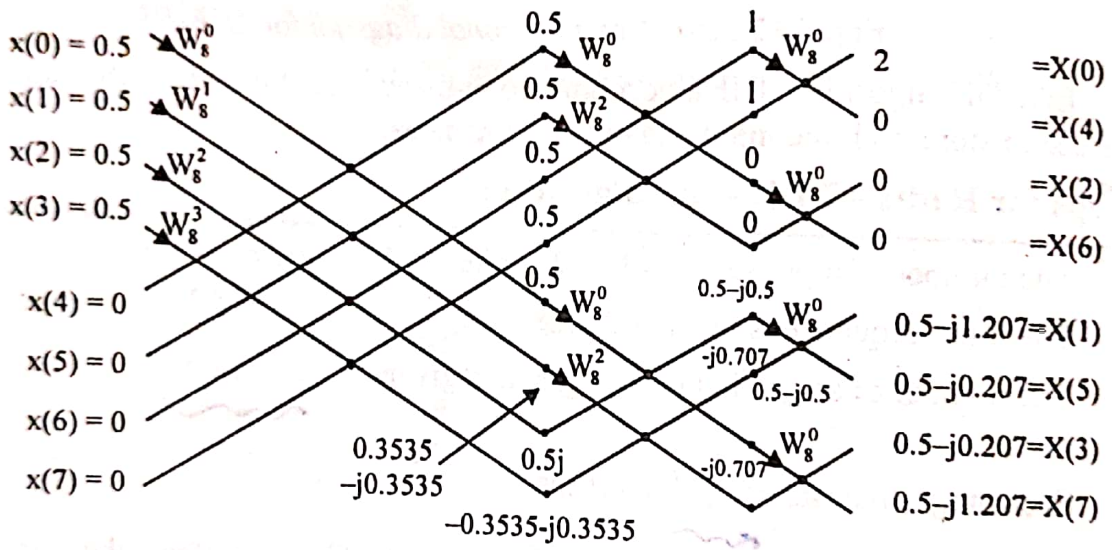
Similarities

Both algorithms require N $\log_2$ N operations to compute the DFT. Both algorithms can be done in-place and both need to perform bit reversal at some place during the computation.

Example 5.3

Compute IDFT of the sequence $X(k) = (7, -0.707, -j, 0.707, 1, 0.707 + j\,0.707, j, -0.707 + j\,0.707\}$ using DIF algorithm.

Solution



$X(k) = \{2, 0.5 - j\,1.207, 0, 0.5 - j\,0.207, 0, 0.5 + j\,0.207, 0, 0.5 + j\,1.207\}$

Example 5.4

Compute 4-point DFT of a sequence $x(n) = \{0, 1, 2, 3\}$ using DIT, DIF algorithm.

Solution

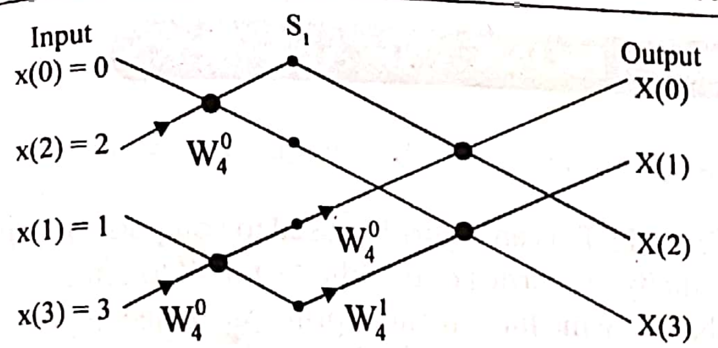
DIT algorithm

Twiddle factors associated with butterflies are

$W_4^0 = 1;\ W_4^1 = e^{-2j\pi/4} = -j$

Bit reversal of input is given by

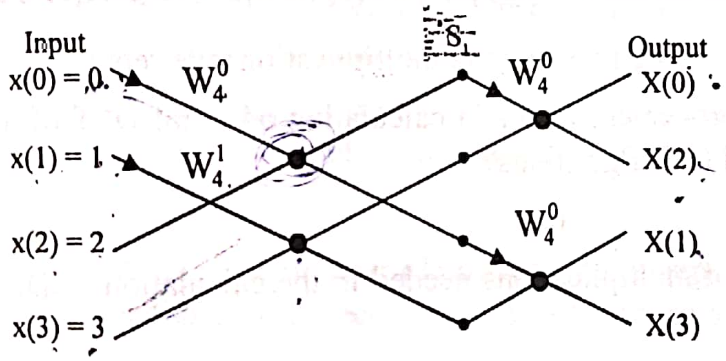
Input index	Binary index	Bit-reversal	Bit-reversal index
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3



Input	S_1	Output
0	$0+2=2$	$2+4=6$
2	$0-2=-2$	$-2+(-j)(-2)=-2+2j$
1	$1+3=4$	$2-4=-2$
3	$1-3=-2$	$-2-(-j)\ (-2)=-2-2j$

$$X(k) = \{6, -2+2j, -2, -2-2j\}$$

DIF



Input	S_1	Output
0	$0+2=2$	$2+4=6$
1	$1+3=4$	$2-4=-2$
1	$0-2=-2$	$-2+2j$
3	$(1-3)(-j)=2j$	$-2-2j$

$$X(k) = \{6, -2+2j, -2, -2-2j\}$$

t=2 means t=0, 1, 2 (In DIT)

t=2 means t=0, ends at 2 (In DIF)

QUESTIONS AND ANSWERS

Q.1 What is FFT?

Ans The fast Fourier transform (FFT) is an algorithm used to compute the DFT. It makes use of the symmetry and periodicity properties of twiddle factor W to effectively reduce the DFT computation time. It is based on the fundamental' principle of decomposing the computation of DFT of a sequence of length N into successively smaller discrete Fourier transforms. The FFT algorithm provides speed-increase factors, when compared with direct computation of the DFT, of; approximately 64 and 205 for 256-point and 1024-point transforms, respectively,

Q.2 Why FFT is needed?

Ans The direct evaluation of DFT using the formula $X(k)=\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$ requires N^2 complex multiplications and N(N — 1) complex additions. Thus for reasonably large values of N (in the order of 1000) direct evaluation of the DFT requires an inordinate amount of computation. By using FFT algorithms the number of computations can be reduced. For example, for an N-point DFT, the number of complex multiplications required using FFT is $\frac{N}{2}\log_2 N$. If N = 16, the number of complex multiplications required for direct evaluation of DFT is 256, whereas using FFT only 32 multiplications are required.

Q.3 What is the speed improvement factor in calculating 64-point DFT of a sequence using direct computation and FFT algorithms?

or

Calculate the number of multiplications needed in the calculation of DFT and FFT with 64-point sequence.

Ans The number of complex multiplications required using direct computation is

$$N^2 = 64^2 = 4096$$

The number of complex multiplications required using FFT is N

$$\frac{N}{2}\log_2 N = \frac{64}{2}\log_2 64 = 192.$$

$$\text{Speed improvement factor} = \frac{4096}{192} = 21.33$$

Q.4 What is the main advantage of FFT?

Ans FFT reduces the computation time required to compute discrete Fourier transform.

Q.5 Calculate the number of multiplications needed in the calculation of DFT using FFT algorithm with 32-point sequence.

Ans For N-point DFT the number of complex multiplications needed using FFT algorithm is $\frac{N}{2}\log_2 N$.

For $N = 32$, the number of complex multiplications is equal to

$$\frac{32}{2}\log_2 32 = 16 \times 5 = 80.$$

Q.6 How many multiplications and additions are required to compute N-point DFT using radix-2 FFT ?

Ans The number of multiplications and additions required to compute N-point DFT using radix-2 FFT are $N\log_2 N$ and $\frac{N}{2}\log_2 N$ respectively.

Q.7 What is meant by radix-2 FFT?

Ans The FFT algorithm is most efficient in calculating N-point DFT. If the number of output points N can be expressed as a power of 2, that is, $N = 2^M$, where M is an integer, then this algorithm is known as radix-2 FFT algorithm,

Q.8 What are the differences and similarities between DIP and DIT algorithms?

Ans ***Differences***

1. For DIT, the input is bit reversed while the output is in natural order, whereas for DIP F the input is in natural order while the output is bit reversed.

2. The DIP F butterfly is slightly different from the DIT butterfly, the difference being that the complex multiplication takes place after the add-subtract operation in DIP.

Similarities

Both algorithms require same number of operations to compute the DFT. Both algorithms can be done in-place and both need to perform bit reversal at. some place during the computation.

Q.9 What is the basic operation of the DIT algorithm?

Ans The basic operation of the DIT algorithm is the so called butterfly in which two inputs $X_m(p)$ and $X_m(q)$ are combined to give the outputs $X_{m+1}(p)$ and $X_{m+1}(q)$ via. the operation

$$X_{m+1}(p) = X_m(p) + W_N^k X_m(q)$$

$$X_{m+1}(p)\,(q) = X_m(p) - W_N^k X_m(q)$$

where W_N^k is twiddle factor.

Q.10 What is the basic operation of the DIF algorithms?

Ans The basic operation of the DIF algorithm is the so called butterfly in which two inputs

$X_m(p)$ and $X_m(q)$ are combined to give the outputs $X_{m+1}(p)$ and $X_{m+1}(q)$ via the operation

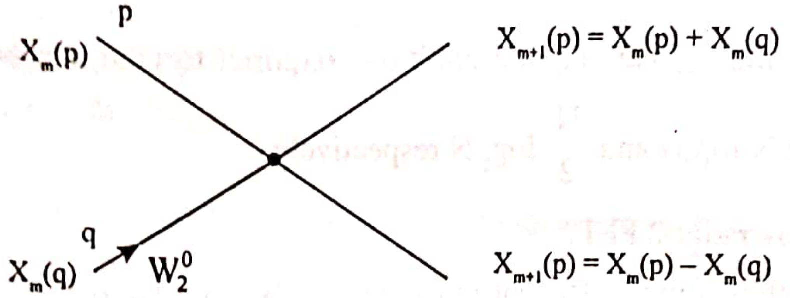
$$X_{m+1}(p) = X_m(p) + X_m(q)$$

$$X_{m+1}(q) = [X_m(p) - X_m(q)]\, W_N^k$$

where W_N^k is twiddle factor.

Q.11 Draw the flow graph of a two-point DFT for a decimation-in-time decomposition.

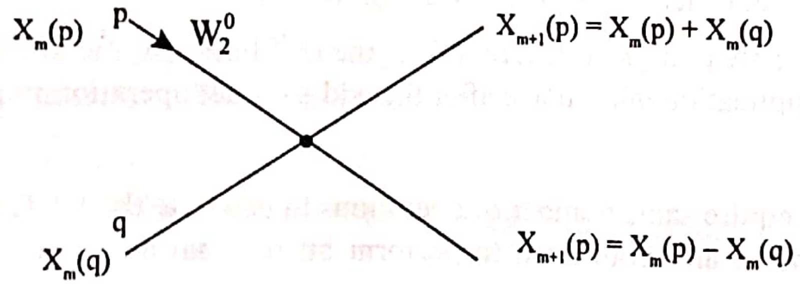
Ans The flow graph of a two-point DFT for a decimation-in-time algorithm is



where $X_m(p)$ and $X_m(q)$ are inputs to the butterfly, $X_{m+1}(p)$ and $X_{m+1}(q)$ are outputs of the butterfly. The nodes p and q represents memory locations.

Q.12 Draw the flow graph of a two-point radix-2 DIF-FFT.

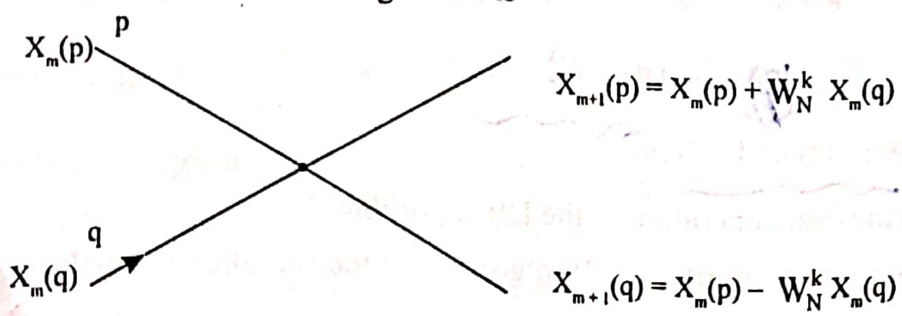
Ans The flow graph of a two-point DFT for a decimation-in-time frequency algorithm is



where $X_m(p)$ and $X_m(q)$ are inputs, $X_{m+1}(p)$ and $X_{m+1}(q)$ are outputs of the butterfly. The nodes p and q represents memory locations.

Q.13 Draw the basic butterfly diagram for DIT algorithm.

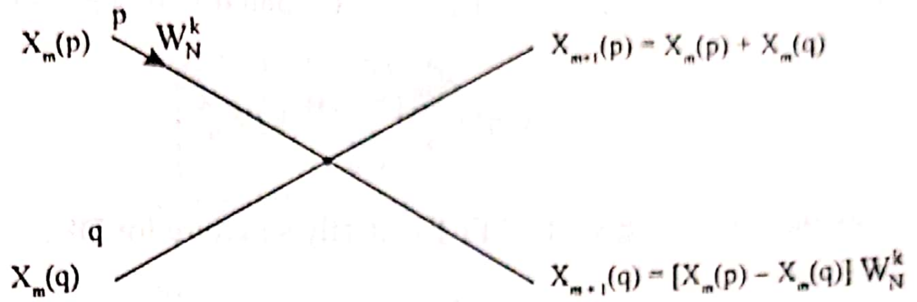
Ans The basic butterfly diagram for DIT algorithm is



where $X_m(p)$ and $X_m(q)$ are inputs to the butterfly, $X_{m+1}(p)$ and $X_{m+1}(q)$ are outputs of the butterfly. The nodes p and q represents memory locations.

Q.14 Draw the basic butterfly diagram for DIP algorithm.

Ans The basic butterfly diagram for DIP algorithm is



where $X_m(p)$ and $X_m(q)$ are inputs, $X_{m+1}(p)$ and $X_{m+1}(q)$ are outputs of the butterfly. The nodes p and q represents memory locations.

Q.15 What is meant by 'm-place' in DIT and DIF algorithms?

Ans The basic butterfly diagrams used in DIT and DIF algorithms are shown in Fig. 1 and Fig. 2 respectively.

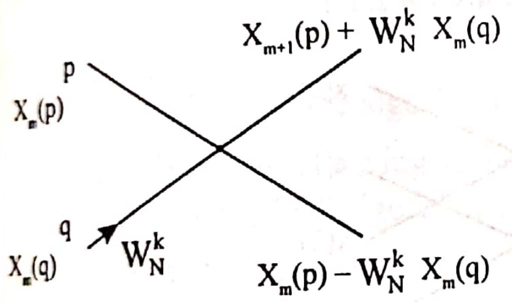


Fig.1

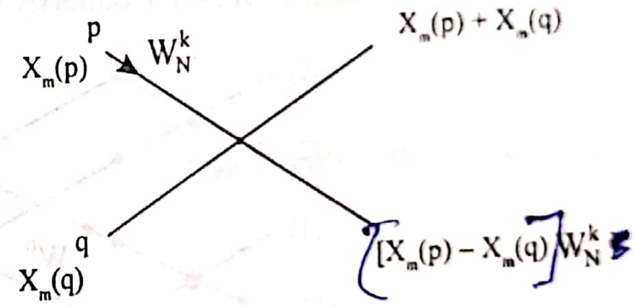


Fig.2

In the Fig. 1 two lines emerging from two nodes cross each other and connected to two nodes on the right hand side. These nodes represents memory locations. At the input nodes $X_m(p)$ and $X_m(q)$, the inputs are stored. After the outputs $X_{m+1}(p)$ and $X_{m+1}(q)$ are calculated, the same memory location is used to store the new values in place of the input values. An algorithm that use the, same location to store both the input and output sequences is called an 'in-place' algorithm.

Q.16 How we can calculate IDFT using FFT algorithm ?

Ans The inverse DFT of an N-point sequence X(k); k = 0, 1... N – 1 is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk} \qquad \text{... (1)}$$

If we take complex conjugate and multiply by N, we get

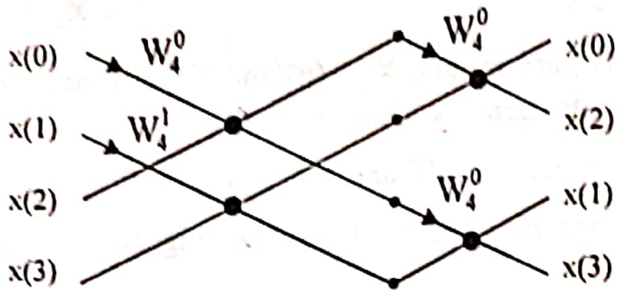
$$Nx^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \quad \text{... (2)}$$

The right hand side of the above equation is DFT of the sequence $X^*(k)$ and may be computed using any FFT algorithm. The desired output sequence x(n) can then be obtained by complex conjugating the DFT of Eq. (2) and dividing by N to give.

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk}\right]^*$$

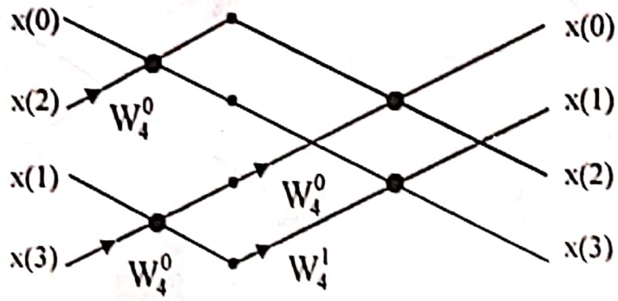
Q.17 Draw the 4-point radix 2 DIF-FFT butterfly structure for DFT.

Ans



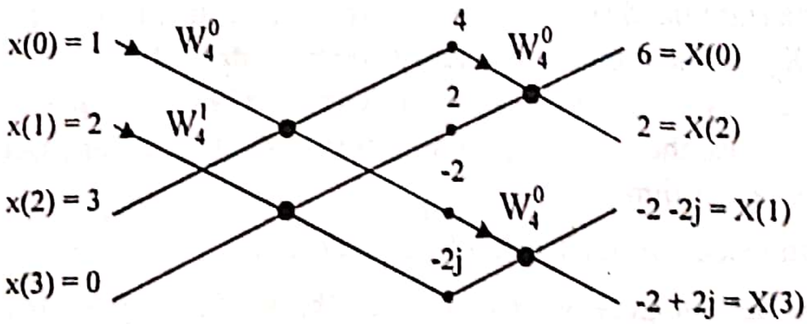
Q.18 Draw the 4-point radix-2 DIT-FFT butterfly structure for DFT.

Ans



Q.19 Find DFT of the sequence x(n) = {1, 2, 3, 0} using DIF algorithm.

Ans



The twiddle factors are $W_4^0 = 1$; $W_4^1 = e^{-j2\pi/4} = -j$

X(k)= $\{6, -2, -2j, 2, -2+2j\}$

Q.20 What are the applications of FFT Algorithms ?

Ans The applications of FFT algorithms includes.

(i) Linear filtering, (ii) Correlation, (iii) Spectrum analysis.

EXERCISE

1. Write the equations and draw the signal flow graph for the decimation in frequency algorithm for N =4.
2. Draw the signal flow graph of decimation-in-time algorithm for N = 8.
3. Compute the DFT for N = 4 if

$$x(n) = 1 \qquad 0 \le n \le 3$$

using the decimation-in-frequency algorithm.

4. Compute the DFT of the sequence for N = 4 if

$$x(n) = \sin\frac{n\pi}{2}$$

using decimation-in-time algorithm.

5. Find the DFT of the following sequences using decimation-in-time (DIT) and decimation-in-frequency (DIF) FFT algorithms.

(a) $s(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

(b) $s(n) = \{1, 0, 0, 0, 1, 1, 1, 0\}$

(c) $s(n) = \{1, 0, 0, 1, -1, 1\}$

(d) $s(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

□□□